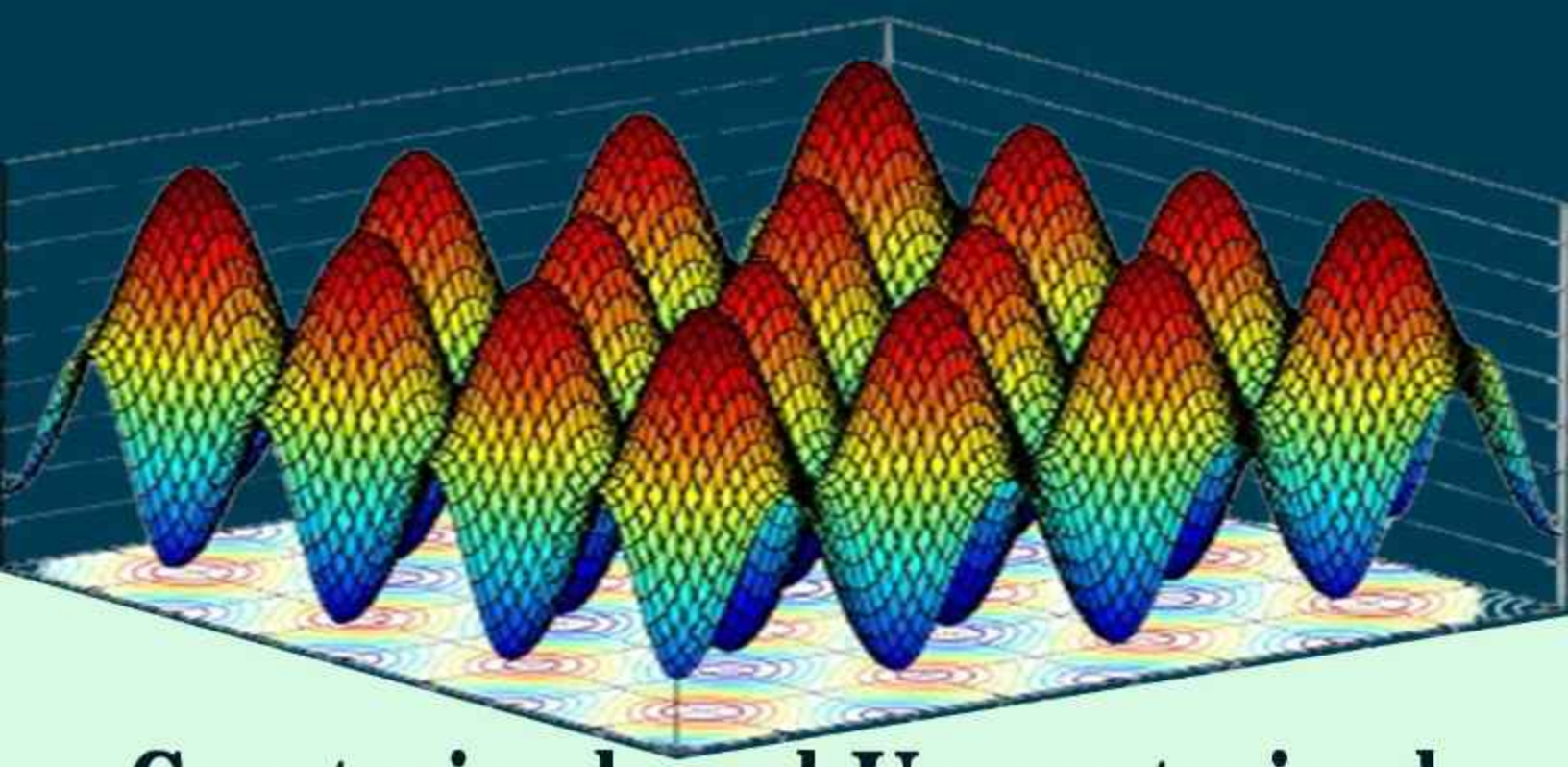
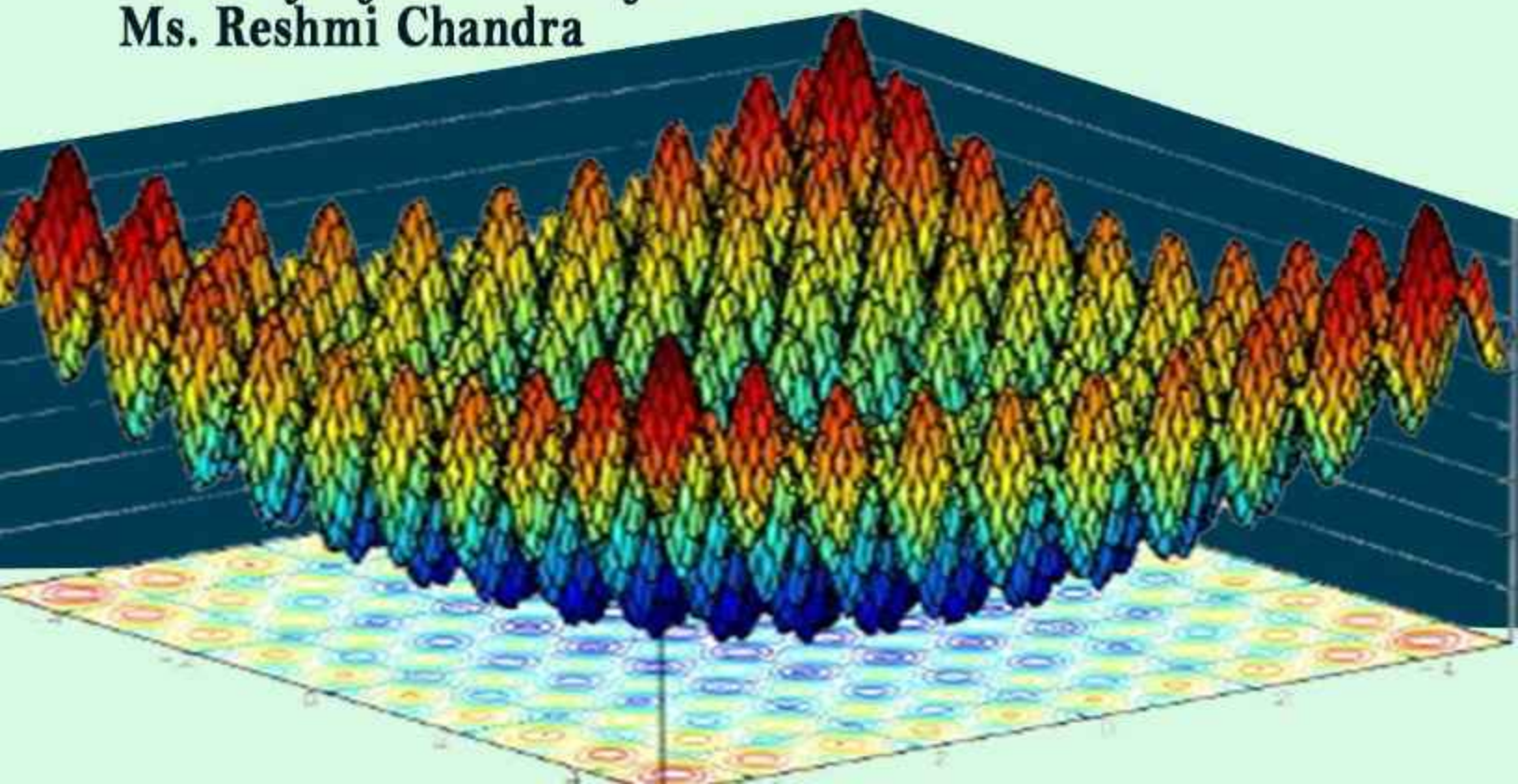


Constrained and Unconstrained Optimization Solutions in Power System

Dr. Bishaljit Paul
Ms. Dipu Mistry
Ms. Priyanjali Mukherjee
Ms. Reshmi Chandra



Constrained and Unconstrained Optimization Solutions in Power System

Authors

Dr. Bishaljit Paul

Ms. Dipu Mistry

Ms. Priyanjali Mukherjee

Ms. Reshmi Chandra



Learnet Publishing

We value, we create

Title of the Book: Constrained and Unconstrained Optimization Solutions in Power System

Authors: Dr. Bishaljit Paul, Ms. Dipu Mistry, Ms. Priyanjali Mukherjee and Ms. Reshmi Chandra

Copyright: Dr. Bishaljit Paul, Ms. Dipu Mistry, Ms. Priyanjali Mukherjee and Ms. Reshmi Chandra

Originality, Responsibility and Disclaimer: The originality, authenticity, and accuracy of the content are entirely the responsibility of the respective author(s). The Publisher shall not be held liable for any form of plagiarism, copyright violation, or legal disputes arising from the content. All opinions expressed are solely those of the author(s) and do not represent the views of the Publisher .

ISBN: 978-81-977844-1-5

MRP: INR 799 (Hard Copy)

INR 499 (Sofy Copy)

Publisher: Learnet Publishing

19/B, Kali Kumar Majumder Road, Post Office-Santoshpur Avenue, Police Station- Survey Park, Kolkata-700075, West Bengal

Email ID: learnetpublishing@gmail.com

info@learnetpub.co.in

Websites: www.learnetpub.co.in

www.jctmg.in

Imprint: Learnet Publishing

PREFACE

Optimization plays a pivotal role in enhancing the efficiency and reliability of modern power systems. Constrained and Unconstrained Optimization Solutions in Power System delves into the mathematical foundations and practical applications of optimization techniques, addressing challenges faced by engineers and researchers in this dynamic field. This book bridges theory and practice, offering insights into solving complex power system problems such as economic dispatch, unit commitment, load flow analysis, and renewable integration.

Designed for academics, professionals, and students, the text explores constrained optimization methods like linear programming and dynamic programming alongside unconstrained techniques such as gradient-based methods. Real-world case studies and computational examples enrich the reader's understanding, ensuring applicability across diverse scenarios.

By blending rigorous analysis with practical implementation, this book aspires to empower its readers to tackle optimization challenges head-on, driving innovation and sustainability in power systems. We hope this work serves as a valuable resource in your journey.

Contents

Chapter 1 Introduction 1
1.1 Structure of a Generic Electric Power System 1
1.2 Power System Models 3
1.3 Power System Control 5
1.4 Power System Security Assessment 8
1.5 Power System Optimization as a Function of Time 11
1.6 Review of Optimization Techniques Applicable to Power Systems 13
References 16

Chapter 2 Electric Power System Models 17
2.1 Introduction 17
2.2 Complex Power Concepts 18
2.3 Three-Phase Systems 20
2.3.1 Y-Connected Systems 21
2.3.2 Delta-Connected Systems 23
2.3.3 Power Relationships 25
2.4 Per Unit Representation 27
2.5 Synchronous Machine Modeling 28
2.5.1 Classical Representation of the Synchronous Machine 29
2.6 Reactive Capability Limits 30
2.7 Prime Movers and Governing Systems 31
2.7.1 Hydraulic Turbines and Governing Models 31
2.7.2 Steam Turbines and Governing System Models 33
2.8 Automatic Gain Control 34
2.8.1 Power Control in a Multigenerator Environment 34
2.8.2 AGC System Models 38
2.8.2.1 Case A: Two Generating Units 38
2.9 Transmission Subsystems 40
2.10 Y-Bus Incorporating the Transformer Effect 42
2.10.1 Fixed Tap-Setting Transformer 42
2.10.2 TCUL Transformer 45
2.10.3 Phase-Shifting Transformer 45
2.11 Load Models 50
2.11.1 Static Load Models 50
2.12 Available Transfer Capability 52
2.12.1 ATC Definition and Formulation 52
2.12.2 ATC Calculation 53

2.13 Illustrative Examples 54
2.14 Conclusions 58
2.15 Problem Set 58
References 61

Chapter 3 Power-Flow Computations 63
3.1 Introduction 63
3.2 Types of Buses for PF Studies 64
3.3 General Form of the PFEs 66
3.3.1 PF Control by Transformer Regulation 67
3.4 Practical Modeling Considerations 69
3.4.1 Generation Subsystem 69
3.4.1.1 Rectangular Formulation 72
3.4.1.2 Polar Formulation 72
3.5 Iterative Techniques for PF Solution 72
3.5.1 G–S Iterative Technique 73
3.5.1.1 G–S Algorithm 73
3.5.1.2 G–S Method Applied to the PFEs 74
3.5.1.3 G–S Iterative Technique 75
3.5.1.4 Line Flow and Losses 77
3.5.2 N–R Method 78
3.5.2.1 N–R Algorithm in the Scalar Case 78
3.5.2.2 N–R Algorithm in the *n*-Dimensional Case 83
3.5.2.3 N–R Algorithm Applied to the PFEs 84
3.5.3 Fast-Decoupled PF Method 97
3.5.4 Linearized (DC) PF Method 100
3.6 Practical Applications of PF Studies 102
3.6.1 Case Study Discussion 103
3.7 Illustrative Examples 103
3.8 Conclusion 108
3.9 Problem Set 109
References 112

Chapter 4 Constrained Optimization and Applications 113
4.1 Introduction 113
4.2 Theorems on the Optimization of Constrained Functions 114
4.2.1 Continuity Assumption 115
4.2.2 Theorems 115
4.3 Procedure for Optimizing Constrained Problems (Functions) 116
4.4 Karush–Kuhn–Tucker Condition 117

4.5 Illustrative Problems 118
4.5.1 Nonpower Systems Application Examples 119
4.6 Power Systems Application Examples 120
4.6.1 Optimal Operation of an All-Thermal System: Equal Incremental Cost-Loading 120
4.6.2 Optimal Operation of an All-Thermal System, Including Losses 123
4.7 Illustrative Examples 127
4.8 Conclusion 133
4.9 Problem Set 134
References 136

Chapter 5 Linear Programming and Applications 137
5.1 Introduction 137
5.2 Mathematical Model and Nomenclature in LP 138
5.2.1 Implicit Assumptions in LP 139
5.3 LP Solution Techniques 140
5.3.1 Graphical Method 140
5.3.2 Matrix Approach to LP 142
5.3.3 Simplex Method 144
5.3.4 Lemma of Matrix Inversion 146
5.3.5 Revised Simplex Method 150
5.4 Duality in LP 153
5.5 Khun–Tucker Conditions in LP 156
5.5.1 Case 1: LP and KKT Conditions for Problems with Equality Constraints 156
5.5.2 Case 2: KKT Applied to the Dual LP Problem 158
5.5.3 Case 3: KKT Applied to LP Problems with Equality Constraints 160
5.6 Mixed-Integer Programming 162
5.6.1 Branch-and-Bound Technique for Binary Integer Programming Problems 164
5.7 Sensitivity Methods for Postoptimization in LP 168
5.7.1 Case 1: Perturbation in the Parameters b_1 169
5.7.2 Case 2: Perturbation in the Cost Coefficients c_j 169
5.7.3 Case 3: Perturbation in the Coefficient a_{ij} 170
5.7.4 Case 4: Injection of New Constraints 170
5.7.5 Case 5: Injection of New Variables 170
5.7.6 Sensitivity Analysis Solution Technique for Changes in Parameters b_i 170
5.7.6.1 Solution Methodology 172
5.7.6.2 Implementation Algorithm 174
5.7.6.3 Duality in Postoptimal Analysis 177

5.8 Power Systems Applications 179
5.9 Illustrative Examples 180
5.10 Conclusion 188
5.11 Problem Set 189
References 196

Chapter 6 Interior Point Methods 197
6.1 Introduction 197
6.2 Karmarkar's Algorithm 199
6.3 Projective-Scaling Method 200
6.4 Dual Affine Algorithm 202
6.5 Primal Affine Algorithm 203
6.6 Barrier Algorithm 204
6.7 Extended IP Method for LP Problems 205
6.8 FI Sequence 206
6.8.1 Optimality Condition 209
6.9 Extended Quadratic Programming Using IP Method 211
6.10 Illustrative Examples 216
6.11 Conclusions 227
6.12 Problem Set 228
References 230

Chapter 7 Nonlinear Programming 233
7.1 Introduction 233
7.2 Classification of NLP Problems 233
7.2.1 NLP Problems with Nonlinear Objective Function and Linear Constraints 233
7.2.2 Quadratic Programming 234
7.2.3 Convex Programming 234
7.2.4 Separable Programming 234
7.3 Sensitivity Method for Solving NLP Variables 235
7.3.1 Procedure for Solving the NLP Problem 236
7.4 Algorithm for Quadratic Optimization 240
7.5 Illustrative Example (Barrier Method for Solving NLP) 241
7.5.1 Algorithm for Recursive Process 242
7.5.1.1 Analytical Forms 245
7.5.1.2 Penalty Vectors 246
7.5.2 Computer Implementation 247
7.6 Illustrative Examples 248
7.7 Conclusion 257
7.8 Problem Set 257
References 262

Chapter 8 Dynamic Programming 263
8.1 Introduction 263
8.2 Formulation of a Multistage Decision Process 264
8.2.1 Representation of a Multistage Decision Process 264
8.2.2 Types of Multistage Decision Problems 266

8.3 Characteristics of DP 266
8.4 Concept of Suboptimization and the Principle of Optimality 267
8.5 Formulation of DP 269
8.6 Backward and Forward Recursion 274
8.6.1 Minimum Path Problem 275
8.6.2 Single Additive Constraint and Additively Separable Return Problem 279
8.6.3 Single Multiplicative Constraint, Additively Separable Return Problem 280
8.6.4 Single Additive Constraint, Multiplicatively Separable Turn Problem 283
8.7 Computational Procedure in DP 283
8.8 Computational Economy in DP 285
8.9 Systems with More than One Constraint 285
8.10 Conversion of a Final Value Problem into an Initial Value Problem 288
8.11 Illustrative Examples 289
8.12 Conclusions 296
8.13 Problem Set 297
References 301

Chapter 9 Lagrangian Relaxation 303
9.1 Introduction 303
9.2 Concepts 304
9.3 Subgradient Method for Setting the Dual Variables 305
9.4 Setting t_k 313
9.4.1 Case 1: Subgradient Method with $t_k = 1$ for All k 313
9.4.2 Case 2: Subgradient Method with $t_k = 1, 0.5, 0.25, \ldots$ 314
9.4.3 Case 3: Subgradient Method with $t_k = 1, 1/3, 1/9, \ldots$ 315
9.5 Comparison with LP-Based Bounds 317
9.6 Improved Relaxation 318
9.7 Summary of Concepts 319
9.8 Past Applications 321
9.9 Summary 322
9.9.1 Overview 322
9.9.2 Algorithm of Solution Using Lagrangian Relaxation Approach 323
9.9.3 Power System Application: Scheduling in Power Generation Systems 324
9.9.3.1 Model 324
9.9.3.2 Relaxation and Decomposition of the Model 326
9.9.3.3 Solution Technique 328
9.10 Illustrative Examples 329
9.11 Conclusions 330
9.12 Problem Set 332
References 333

1

Introduction

1.1 Structure of a Generic Electric Power System

Although no two electric power systems are alike, all share some common fundamental characteristics including the following:

1. Electric power is generated using synchronous machines that are driven by turbines (steam, hydraulic, diesel, or internal combustion).
2. Generated power is transmitted from the generating sites over long distances to load centers that are spread over wide areas.
3. Three-phase AC systems comprise the main means of generation, transmission, and distribution of electric power.
4. Voltage and frequency levels are required to remain within tight tolerance levels to ensure a high-quality product.

The basic elements of a generic electric power system [1,3] are displayed in Figure 1.1. Electric power is produced at generating stations (GS) and transmitted to consumers through an intricate network of apparatus including transmission lines, transformers, and switching devices.

A transmission network is classified as

1. Transmission system
2. Subtransmission system
3. Distribution system

The transmission system interconnects all major GS and main load centers in the system. It forms the backbone of the integrated power system and operates at the highest voltage levels (typically, 230 kV and above). The generator voltages are usually in the range of 11–35 kV. These are stepped up to the transmission voltage level, and power is transmitted to transmission substations where the voltages are stepped down to the subtransmission

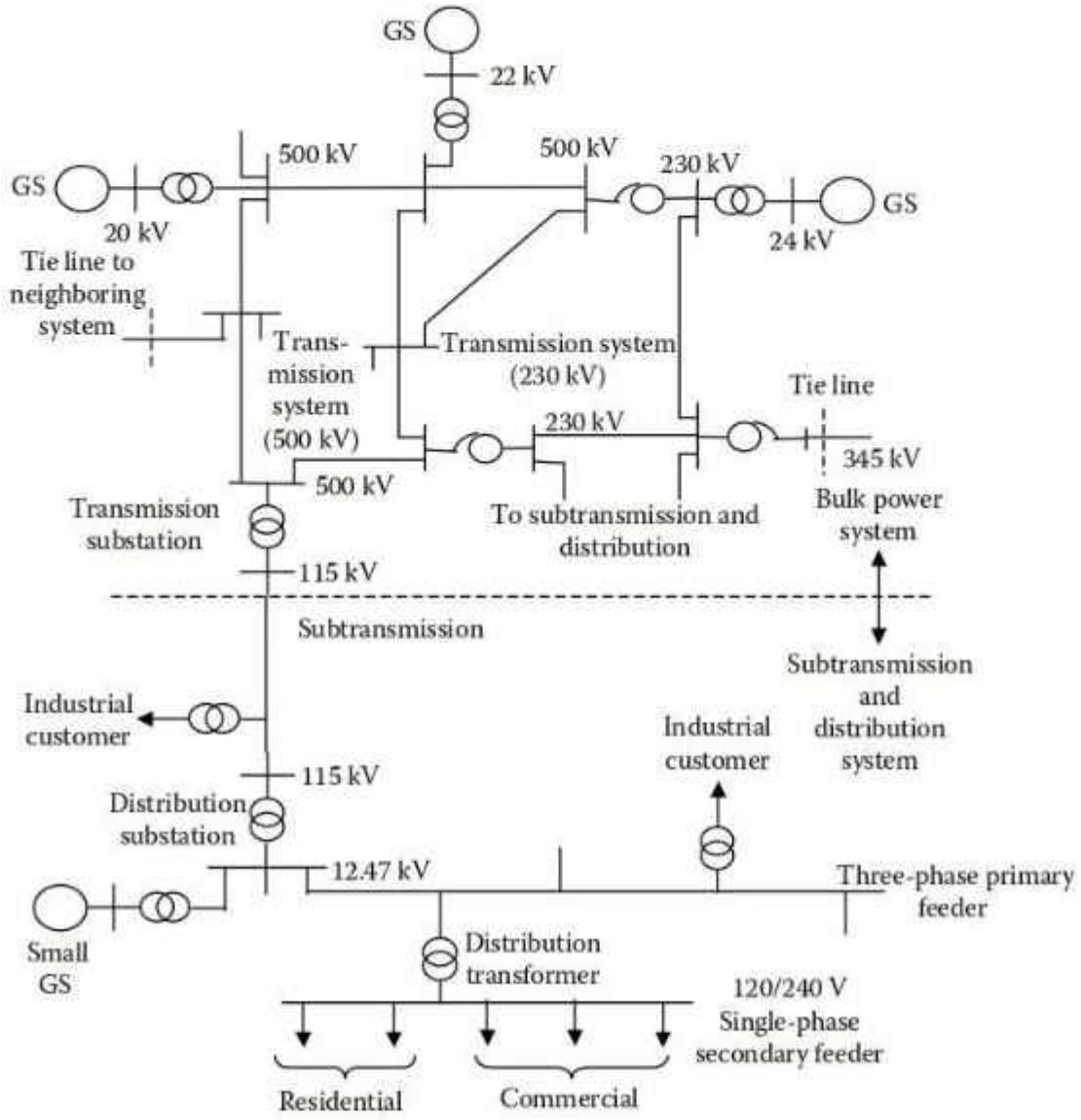


FIGURE 1.1
Basic elements of a power system.

level (typically, 69–138 kV). The generation and transmission subsystems are often referred to as the bulk power system.

The subtransmission system transmits power at a lower voltage and in smaller quantities from the transmission substation to the distribution substations. Large industrial customers are commonly supplied directly from the subtransmission system. In some systems, as expansion and higher voltage levels become necessary for transmission, the older transmission lines are often relegated to subtransmission function.

The distribution system is the final stage in the transfer of power to the individual customers. The primary distribution voltage is typically between 4.0 and 34.5 kV. Primary feeders at this voltage level supply small industrial customers. The secondary distribution feeders supply residential and commercial customers at 120–240 V.

1.2 Power System Models

In order to be able to control the power system from the point of view of security, one is required to know two basic things: the mathematical models of the system and the variety of control functions and associated objectives that are used. In this section some general remarks about power system models are given.

In Figure 1.2, we show the basic decomposition of the system into a set of generators that deliver electrical power to the distributed load by means of a network. In our subsequent discussion we start by describing the load, then the network, and finally the generators.

In assessing load behavior as seen from a substation, one is interested typically in items such as the following:

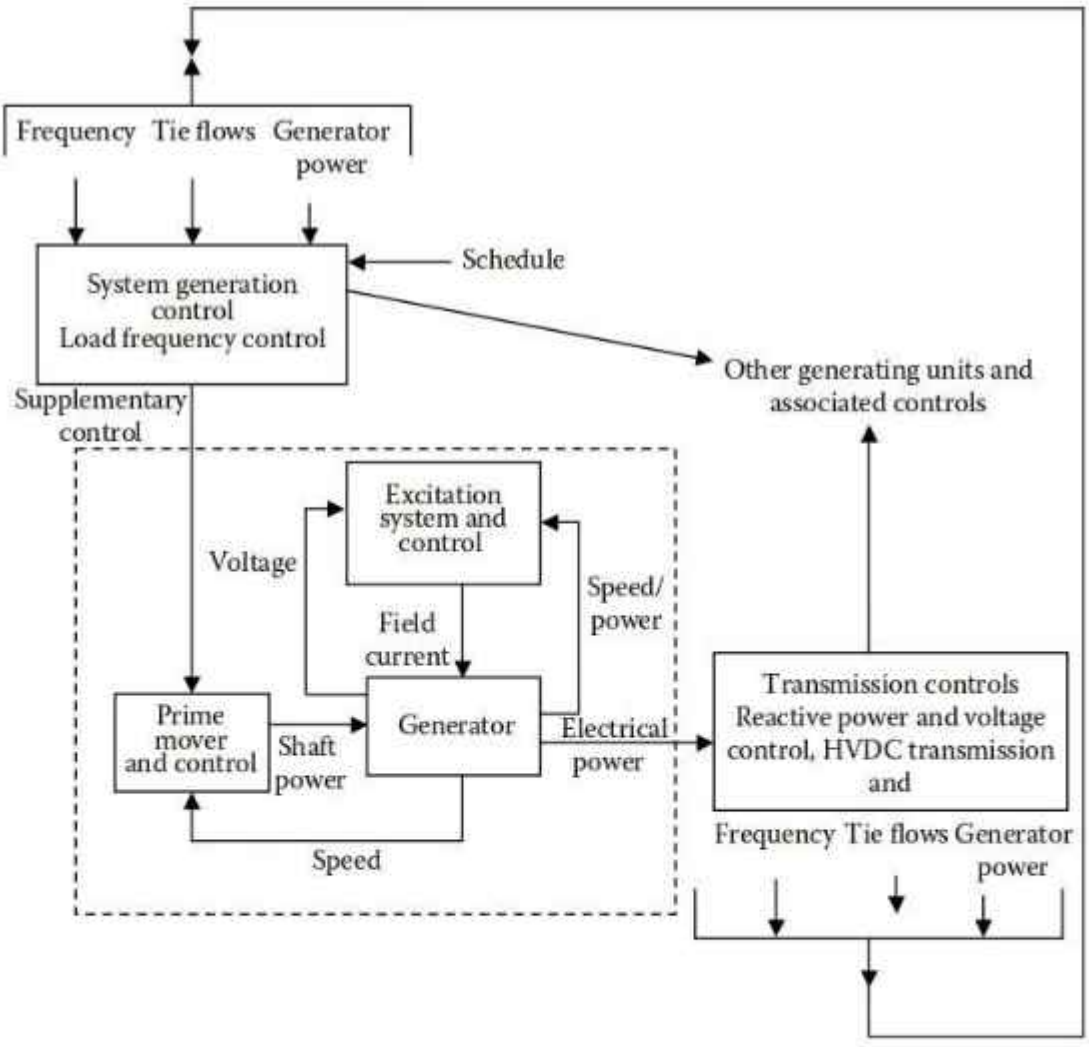


FIGURE 1.2
Subsystems of a power system and associated controls.

- Present value of real power consumed in megawatt (MW) and the associated power factor (or reactive power)
- Forecast values of real and reactive power over a range of future times—next few minutes, to days and years
- Load response characteristics (e.g., lumped circuit representation or transfer function) for fluctuations in substation voltage and frequency

By knowing the real and reactive components of substation loads (present or forecast), one can establish a complete picture for a steady-state bulk power demand in the system. Furthermore, by identifying load response characteristics one can adequately evaluate the dynamic behavior of the demand in the presence of disturbances and feedback controllers.

The network portion of the system consists of the transmission, subtransmission, and distribution networks. This division is based on voltage levels, and, consequently, on the ratings of various circuits. Typically, power transmission is done at voltages that can range from 115 to 765 kV. The transmission network is not necessarily radial. In fact, it has many closed loops as required for reliable supply purposes. Subtransmission (65–40 kV) and distribution (20–115 kV) systems are primarily radial when operated. Because of this arrangement, analysis at the bulk power levels considers only the transmission portion of the network. From a transmission substation, real and reactive power will flow radially to the load through a sequence of step-down transformers and power lines.

In modeling the network elements, one should identify the type of problem being analyzed. Under normal conditions, the load fluctuates very slowly compared to transient time constants associated with transmission lines. And since system frequency is maintained at its nominal value quite accurately, the lumped circuit representation of transmission lines is quite adequate. On the other hand, if electromagnetic transients resulting, for example, from lightning strikes are being investigated, then wave equations should be considered. In our present context, the lumped circuit representation can be very adequate.

In a block diagram of a typical generator, the blocks correspond to the main components of the power plant. The significant outputs of the generator as measured at the terminals are its

Voltage magnitude (kV)

Real power produced (MW)

Reactive power produced (MVAr)

Speed (in radians and denoted by $w = 2\pi f$, where f is the frequency)

Under normal conditions, three of those quantities are continuously controlled by the power plant. These are the terminal voltage, the frequency (speed), and real power output. The output voltage V is subtracted from the

specified voltage V_0 and the difference is an error signal that drives the exciter system. The exciter, in turn, modifies the field voltage of the turbo-alternator in such a way that V becomes closer in value to V_0. The same feedback concept applies to the control of frequency and real power. In this case, however, the corresponding error signal drives the governor system. The governor controls valve openings of the turbine, which in turn control its speed and mechanical power output. The power error signal can also go back to the prime mover (boiler, in the case of steam generation), so that more, or less, steam is generated.

The exciter normally has a fast response (10^{-2}–10^{-1} s). The governor-turbine system is slower (0.1–1 s) in its response. However, since the load is much slower in its changes than the response times, it is safe to assume that perfect control is always present; that is, normal operation is, to a high degree, sinusoidal steady-state operation. Only when events that are fast relative to governor-turbine or exciter response times, one would worry about the steady-state operation. Thus, in the presence of network faults, or immediately following switching operations, one needs to consider transient or dynamic representation of the system.

1.3 Power System Control

Before discussing how the system is controlled, one needs to briefly summarize the means by which control action is obtained.

First, let us understand the meaning of control as it applies to the power system case. The system is normally designed so that certain quantities can be manipulated by means of devices. Some of these are the so-called status quantities. By means of power circuit breakers, a transmission line is open (status = OFF) or dosed (status = ON). Some of them are integer variables (tap-settings on power transformers). And the rest are continuous variables such as the real power output of a generator. The control devices can be simple, like fuses, or highly complex dynamic systems, like exciters and governors.

Control action is attained by the manipulation of all control devices that exist on the system. This is achieved in order to meet different, but consistent objectives, and through a variety of means. The general objectives of system control are listed in order as follows:

1. Protection of major pieces of equipment and of system integrity
2. Continuity of high-quality service
3. System secure operation
4. System economic and environmentally acceptable operation
5. Emergency state control
6. Restorative control in minimum time

As a rule, control action is based on information derived from direct measurements and inferred data. Each control device will require certain kinds of information based on the following considerations:

Speed-of-response requirements

Impact of control action (i.e., global vs. local)

Relative importance of different pieces of information (e.g., local vs. distant information)

Some examples of this are now in order. For a short-circuit fault on a transmission line, the main objective is to protect the system from going unstable (losing synchronism) and to protect the line from serious damage. This is achieved by correct breaker action, which will open that line and isolate it from the system. Normally, however, other neighboring lines and transformers feel the effect of the short circuit. Hence, it is important to open the faulted line first. By means of offline short-circuit analysis, relay settings are established so that the faulted line will open first. Hence, the only needed online information for that purpose is line current. This is strictly local information.

In a more complicated situation, we can look at the problem of maintaining a satisfactory voltage profile in the system. Scheduled generator terminal voltage is attained by means of local feedback control to the exciter. The values of scheduled voltages, which are the set points in the feedback loop, are established from an analysis of the entire system's operating conditions. In most cases, offline analysis of the system yields values of scheduled voltages. Modern, energy control centers (ECC) have the capability of processing global online information and updating voltage profile set points [5].

The function of an electric power system is to convert energy from one of the naturally available forms to the electrical form and to transport it to the points of consumption. Energy is seldom consumed in the electrical form but is rather converted to other forms such as heat, light, and mechanical energy. The advantage of the electrical form of energy is that it can be transported and controlled with relative ease and with a high degree of efficiency and reliability. A properly designed and operated power system should, therefore, meet the following fundamental requirements:

1. It must be able to meet the continually changing load demand for active and reactive power. Unlike other types of energy, electricity cannot be conveniently stored in sufficient quantities. Therefore, adequate spinning reserve of active and reactive power should be maintained and appropriately controlled at all times.
2. It should supply energy at minimum cost and with minimum ecological impact.

3. The "quality" of the power supply must meet certain minimum standards with regard to the factors:
   a. Constancy of frequency
   b. Constancy of voltage
   c. Level of reliability

Several levels of controls involving a complex array of devices are used to meet the above requirements. These are depicted in Figure 1.2, which identifies the various subsystems of a power system and the associated controls. In this overall structure, there are controllers operating directly on individual system elements. In a generating unit these consist of prime mover controls and excitation controls. The prime mover controls are concerned with speed regulation and control of energy supply system variables such as boiler pressures, temperatures, and flows. The function of the excitation control is to regulate generator voltage and reactive power output. The desired MW outputs of the individual generating units are determined by the system generation control.

The primary purpose of the system generation control is to balance the total system generation against system load and losses so that the desired frequency and power interchange with neighboring systems (tie flows) is maintained.

The transmission controls include power and voltage control devices, such as static VAr compensators, synchronous condensers, switched capacitors and reactors, tap-changing transformers, phase-shifting transformers, and high voltage direct current (HVDC) transmission controls.

These controls described above contribute to the satisfactory operation of the power system by maintaining system voltages and frequency and other system variables within their acceptable limits. They also have a profound effect on the dynamic performance of the power system and its ability to cope with disturbances.

The control objectives are dependent on the operating state of the power system. Under normal conditions, the control objective is to operate as efficiently as possible with voltages and frequency close to nominal values. When an abnormal condition develops, new objectives must be met to restore the system to normal operation.

Major system failures are rarely the result of a single catastrophic disturbance causing collapse of an apparently secure system. Such failures are usually brought about by a combination of circumstances that stress the network beyond its capability. Severe natural disturbances (such as a tornado, severe storm, or freezing rain), equipment malfunction, human error, and inadequate design combine to weaken the power system and eventually lead to its breakdown. This may result in cascading outages that must be contained within a small part of the system if a major blackout is to be prevented.

Protecting isolated systems has been a relatively simple task, which is carried out using overcurrent directional relays with selectivity being obtained by time grading. High-speed relays have been developed to meet the increased short-circuit currents due to the larger size units and the complex interconnections.

For reliable service, an electric power system must remain intact and be capable of withstanding a wide variety of disturbances. It is essential that the system be operated so that the more probable contingencies can be sustained without loss of load (except that connected to the faulted element) and so that the most adverse possible contingencies do not result in widespread and cascading power interruptions.

The November 1965 blackout in the northeastern part of the United States and Ontario had a profound impact on the electric utility industry. Many questions were raised and led to the formation of the National Electric Reliability Council in 1968. The name was later changed to the North American Electric Reliability Council (NERC). Its purpose is to augment the reliability and adequacy of bulk power supply in the electricity systems of North America. The NERC is composed of nine regional reliability councils and encompasses virtually all the power systems in the United States and Canada. Each regional council has established reliability criteria for system design and operation. Since differences exist in geography, load pattern, and power sources, criteria for the various regions differ to some extent.

Design and operating criteria play an essential role in preventing major system disturbances following severe contingencies. The use of criteria ensures that, for all frequently occurring contingencies, the system will, at worst, transit from the normal state to the alert state, rather than to a more severe state such as the emergency state or the in extremis state. When the alert state is entered following a contingency, operators can take actions to return the system to the normal state.

1.4 Power System Security Assessment

Power system security is the ability of the bulk power electric power system to withstand sudden disturbances such as electric short circuits or unanticipated loss of system components [5]. In terms of the requirements for the proper planning and operation of the power system, it means that following the occurrence of a sudden disturbance, the power system will

1. Survive the ensuing transient and move into an acceptable steady-state condition.
2. In this new steady-state condition, all power system components operate within established limits.

Electric utilities require security analysis to ensure that, for a defined set of contingencies, the above requirements are met. The analysis required to survive a transient is complex, because of increased system size, greater dependence on controls, and more interconnections. Additional complicating factors include the operation of the interconnected system with greater interdependence among its member systems, heavier transmission loading, and concentration of the generation among few large units at light loads.

After the 1965 blackout, various efforts went into improving reliable system operation. Several reliability criteria and emergency guidelines were introduced by the Federal Power Commission (FPC) and North American Power System Interconnection Committee (NAPSIC). Summaries of these guidelines are given in Ref. [2, Appendix]. These guidelines and criteria represent efforts by the government and the utilities to improve control and operational practices.

More important, however, were the efforts by various researchers and specialists in what has come to be known as the secure control of the power system. In DyLiacco's pioneering work [6], the power system is judged to reside at any instant of time in any of three operating states: normal, emergency, and restorative.

Under normal steady-state operating conditions all customer demands are met and all equipments operate below its rated capacity. Theoretically speaking, the requirement of meeting customer demands is expressed mathematically by means of a set of equations (equality constraints) of the type

$$\begin{aligned} h_1(x_1, \ldots, x_n; u_1, \ldots, u_m) &= 0 \\ h_2(x_1, \ldots, x_n; u_1, \ldots, u_m) &= 0 \\ &\vdots \\ h_n(x_1, \ldots, x_n; u_1, \ldots, u_m) &= 0, \end{aligned} \tag{1.1}$$

where

$x_1, \ldots, x_n$ is a set of dependent (state) variables

$u_1, \ldots, u_m$ is a set of independent (input, demand, or control) variables

Typically these equality constraints correspond to the so-called load-flow equations. The constraints relative to equipment can be written, in general, in the following form:

$$\begin{aligned} g_1(x_1, \ldots, x_n; u_1, \ldots, u_m) &\leq 0 \\ g_2(x_1, \ldots, x_n; u_1, \ldots, u_m) &\leq 0 \\ &\vdots \\ g_l(x_1, \ldots, x_n; u_1, \ldots, u_m) &\leq 0. \end{aligned} \tag{1.2}$$

They correspond to items such as upper and lower limits on power generation by a given unit, current limits on transmission lines and transformers, and so on.

Mathematically, the normal operating state is defined whenever the utility system considered satisfies Equations 1.1 and 1.2.

Following certain disturbance events (short circuits due to faults, loss of generation, loss of load, and others) some of the inequality constraints may be violated. For example, a line may become overloaded, or system frequency may drop below a certain limit. In these cases the system is in the emergency operating state.

Finally, the system may exist in a situation where only a fraction of the customers are satisfied without overloading any equipment. In this case only a portion of the system is in the normal state. As a result, not all the equality constraints are satisfied, but the inequality constraints are. Such a state is called the restorative operating state.

Symbolically, we can rewrite Equations 1.1 and 1.2 in the following form.

$$\left.\begin{aligned} h(x,u) &= 0 \\ g(x,u) &\leq 0 \end{aligned}\right\}. \tag{1.3}$$

With this notation, we summarize our definition of the three operating states as follows:

Normal state

$$h(x,u) = 0$$
$$g(x,u) \leq 0.$$

Emergency state (need to change $g(x,u)$ to not less than or equal to)

$$h(x,u) = 0$$
$$g(x,u) \leq 0.$$

Restorative state

$$h(x,u) \neq 0$$
$$g(x,u) \leq 0.$$

The security of the system is defined relative to its ability to withstand a number of postulated disturbances. A normal state is said to be secure if, following any one of the postulated disturbances, the system remains in the normal state. Otherwise, it is insecure.

In the online operation of the system, one monitors the different variables that correspond to its operating conditions. This monitoring process is called security monitoring. The process of determining whether the system is in the secure normal state is called security assessment.

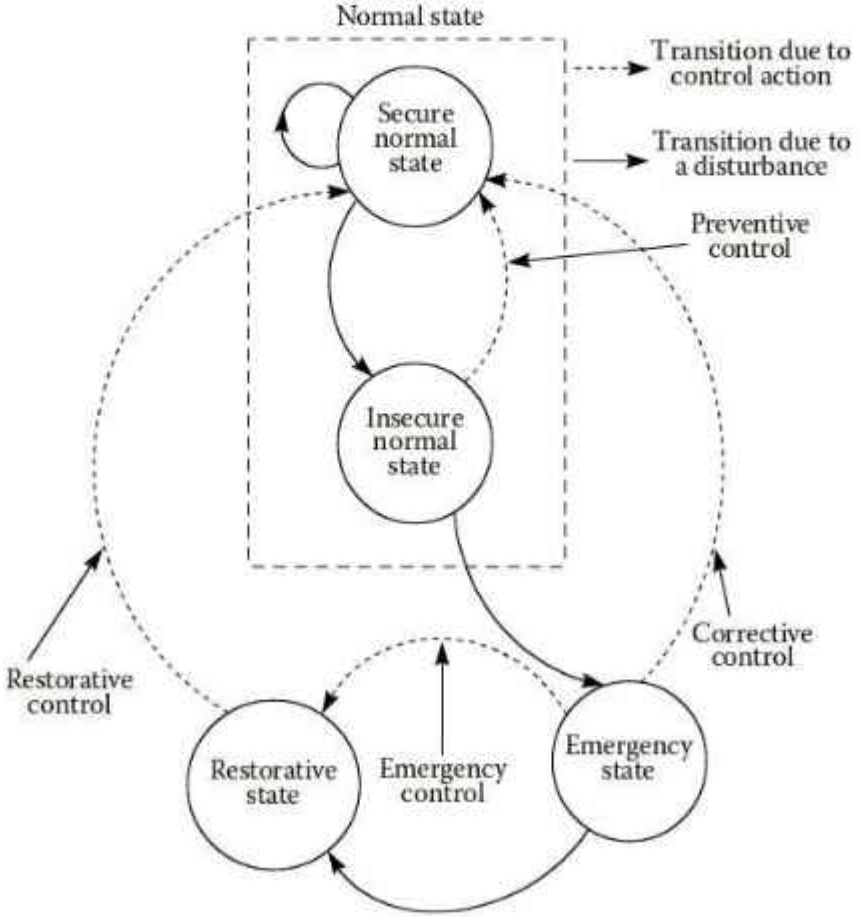


FIGURE 1.3
Operating states of a power system.

In the process of security assessment it may be concluded that the system is in the insecure normal state. In that case, the system operator will attempt to manipulate system variables so that the system is secure. This action is called preventive control. If, on the other hand, the system is in an emergency state, then two types of control action are possible. In the first type, called corrective control, action is possible whereby the system is sent back to the normal state. If corrective control is not possible, then emergency control is applied [6]. This control can be due to relay-initiated action, automatic control, or operator control. In any case, the system will drift to the restorative state as a result of emergency control. Finally, in the restorative state, control action is initiated to restore all services by means of restorative control. This should put the system back to the normal state. Figure 1.3 illustrates the various transitions due to disturbances as well as various control actions.

1.5 Power System Optimization as a Function of Time

The hourly commitment of units, the decision whether a unit is on or off at a given hour, is referred to as unit commitment. Hourly production

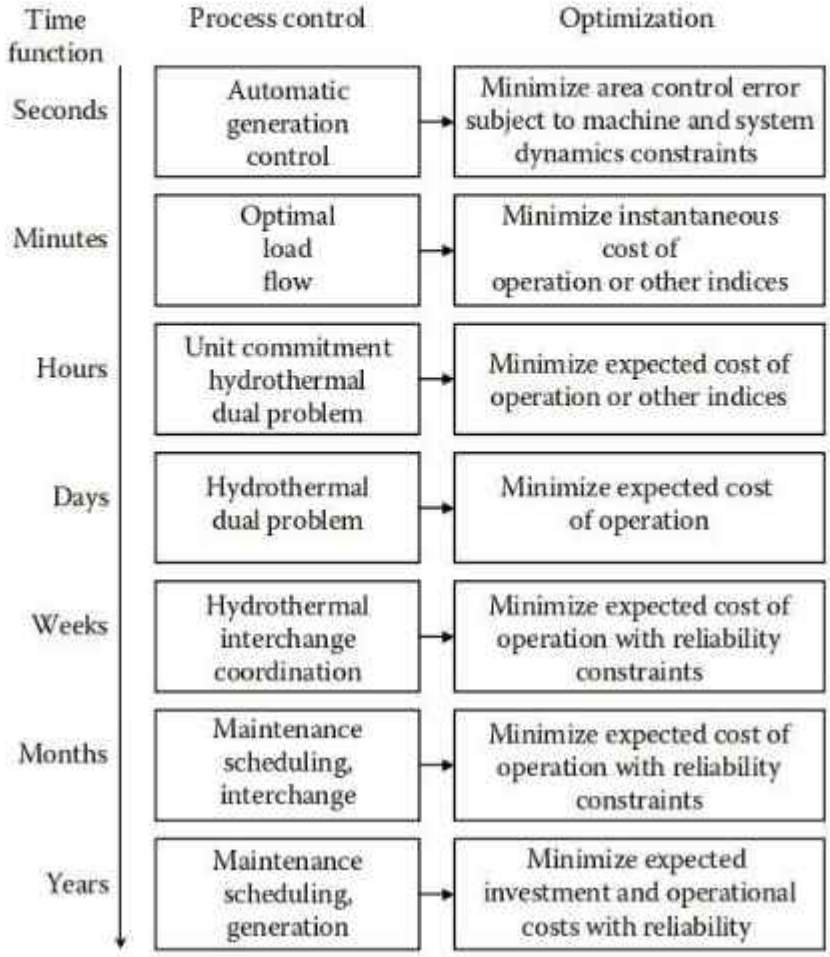


FIGURE 1.4
Time horizon of the power system optimization problem.

of hydroelectric plants based on the flexibility of being able to manage water reserve levels to improve system performance is referred to as the hydrothermal problem and hourly production of coal generation or a dual purpose plant is called the dual purpose problem. Scheduling of unit maintenance without violating reserve capacity while minimizing the production cost is referred to as a maintenance scheduling problem. The interdependence among the various control optimization problems as the time horizon expands from seconds to years is shown in Figure 1.4.

In power system operation and planning, there are many optimization problems that require real-time solutions such that one can determine the optimal resources required at minimum cost within a given set of constraints. This scheduling is done over time (minutes, hours, days, etc.). In this regard, we classify the problem as either operational or planning. Notably, in the operations scheduling problem, we usually extend the studies up to 24 h. On the other hand, planning problems are solved in the time frame of years.

In analyzing the optimization problem, there are many controllable parameters of interests. There are many objective functions and constraints that must be satisfied for economic operation. (These objectives and

constraints are quantified later.) Methods exist for solving the resulting economic dispatch problem as a function of time when we incorporate the constraints of the system and typically the economic dispatch problem evolves. It uses mathematical techniques such as linear programming (LP), unconstrained optimization techniques (using Lagrange multipliers), and nonlinear programming (NLP) to accommodate the constraints [4]. The availability of these techniques in addressing this problem has been noted. Other variations on the economic dispatch problem are hydrothermal and unit commitment problems.

Dynamic programming (DP), Lagrange relaxation technique, and Bender's decomposition algorithm are used to solve this class of optimization problem. Another method in power system operation and control is the optimal maintenance of units and generators.

Finally, in the same realm is the optimal power flow (OPF), which holds the promise of extending economic dispatch to include the optimal setting of under load tap-changers, generator real and reactive powers, phase-shifter taps, and the like. OPF has been expanded as new problems arise to include new objective functions and constraints. And OPF has attracted researchers to the development of new optimization algorithms and tests as a routine base. Other applications extending the work to optimization of the network include VAr planning, network expansion, and availability transfer capability.

At the distribution end, loss minimization, data estimation, and network reconfiguration have demanded optimum decision making as a planning problem as well as an operations problem. There are mathematical optimization techniques ranging from LP to evolutionary search techniques that can be employed to obtain optimum distribution networks.

There is a need to summarize the essential mathematical methods that have been fully developed, tested, and utilized on a routine basis for security analysis of the power system. The selection of the appropriate optimization technique depends on the system as defined by the objective functions and the constraints. The constraints are divided into two classes, namely, technical and nontechnical. The class of technical constraints includes network, equipment, and device constraints. The class of nontechnical constraints includes social, environmental, and economic limitations.

1.6 Review of Optimization Techniques Applicable to Power Systems

In the early days of power system operation, the OPF tool was defined in terms of the conventional economic dispatch problem aimed at determining the optimal settings for control variables in a power system with respect to

various constraints. However, the capability of power system optimization tools has been broadened to provide solutions for a wide range of utility-dependent problems. Today, the OPF tool is used to solve a static constrained nonlinear optimization problem whose development has followed closely the advances in numerical optimization techniques and computer technology. Commonly available OPF packages can solve very large and complex power system formulations in a relatively short time.

Generally, OPF requires solving a set of nonlinear equations, describing optimal and secure operation of a power expressed as

$$\begin{aligned} &\textit{Minimize} && F(\mathbf{x},\mathbf{u}) \\ &\text{while satisfying} && \mathbf{g}(\mathbf{x},\mathbf{u}) = 0 \\ &&& \mathbf{h}(\mathbf{x},\mathbf{u}) \leq 0, \end{aligned}$$

where

g(x,u) is the set of nonlinear equality constraints (power flow equations)
h(x,u) is the set of inequality constraints of vector arguments **x** and **u**
x is the vector of dependent variables consisting of bus voltage magnitudes and phase angles, as well as MVAr loads, fixed bus voltages, line parameters, and so on
u is the vector of control variables

Vector **u** includes the following:

Real and reactive power generation
Phase-shifter angles
Net interchange
Load MW and MVAr (load shedding)
DC transmission line flows
Control voltage settings
Load tap changer (LTC) transformer tap settings

Common objectives in a power system include

Active power cost minimization
Active power loss minimization
Minimum control shift
Minimum number of controls scheduled

Examples of the associated equality and inequality constraints are

Limits on all control variables
Power flow equations

Generation/load balance
Branch flow limits
Bus voltage limits
Active and reactive reserve limits
Generator MVAr limits
Corridor (transmission interface) limits

The optimization methods that are incorporated in the OPF tools can be classified based on optimization techniques such as

LP based methods
NLP based methods
Integer programming (IP) based methods
Separable programming (SP) methods

Notably, LP is recognized as a reliable and robust technique for solving a wide range of specialized optimization problems characterized by linear objectives and linear constraints. Many commercially available power system optimization packages contain powerful LP algorithms for solving power system problems for both planning and operator engineers. LP has extensions in the simplex method, revised simplex method, and interior point techniques.

Interior point techniques are based on the Karmarkar algorithm and encompass variants such as the projection scaling method, dual affine method, primal affine method, and barrier algorithm.

In the case of the NLP optimization methods, the following techniques are introduced:

Sequential quadratic programming (SEQ)
Augmented Lagrangian methods
Generalized reduced gradient method
Projected augmented Lagrangian
Successive LP
Interior point methods

The basic formulation is then extended to include security and environmental constraints, which have become very important factors in power system operation in the past few decades. Special decomposition strategies are also applied in solving large-scale system problems. These include Benders decomposition, Lagrangian relaxation, and Talukdar–Giras optimization techniques.

In recent years, the advancement of computer engineering and the increased complexity of the power system optimization problem have led

to greater need and application of specialized programming techniques for large-scale problems. These include DP, Lagrange multiplier methods, and evolutionary computation methods such as genetic algorithms. These techniques are often hybridized with many other techniques of intelligent systems, including artificial neural networks, expert systems, Tabu-search algorithms, and fuzzy logic.

References

1. Bergen, A. R. and Vittal, V., *Power Systems Analysis*, 2nd edn., Prentice-Hall, Upper Saddle River, NJ, 1999.
2. Elgerd, O. I., *Electrical Energy Systems Theory—An Introduction*, McGraw-Hill, New York, 1982.
3. Gonen, T., *Electric Power Distribution System Engineering*, McGraw-Hill, New York, 1986.
4. Luenberger, D. G., *Introduction to Linear and Nonlinear Programming*, Addison-Wesley, Reading, MA, 1975.
5. Wood, A. J. and Wollenberg, B. F., Ed., *Power Generation, Operation, and Control*, Wiley, New York, 1996.
6. DyLiaco, T. E., *An Overview of Power System Control Centres*, IEEE Tutorial Course on Energy Control Center Design, 1977.

2

Electric Power System Models

2.1 Introduction

The power industry in the United States has been engaged in a changing business environment for quite some time now, moving away from a centrally planned system to one in which players operate in a decentralized fashion with little knowledge of the full state of the network, and where decision making is likely to be market driven rather than based on technical considerations.

The new environment differs markedly from the one in which the system previously has been operated. This leads to the requirement of some new techniques and analysis methods for system operation, operational and long-term planning, and the like.

Electrical power systems vary in size, topography, and structural components. However, the overall system can be divided into three subsystems: generation, transmission, and distribution. System behavior is affected by the characteristics of every major element of the system. The representation of these elements by means of appropriate mathematical models is critical to the successful analysis of system behavior. For each different problem, the system is modeled in a different way. Here, power system under steady constraints has been extended to include load conditions, thermal limits, stability constraints, congestion limits, and other constraints in pricing mechanism, which are introduced in numerous real-life applications discussed in subsequent chapters of this book.

This chapter describes some system models for analysis purposes and introduces concepts of power expressed as active, reactive, and apparent, followed by a brief review of three-phase systems and the per unit system representations (Sections 2.3 and 2.4). Section 2.5 discusses modeling the synchronous machine from an electric network standpoint. Reactive capability curves are examined in Section 2.6 followed by discussion of static and dynamic load models later in Section 2.11. We now introduce some fundamental concepts and background knowledge required in understanding power systems analysis.

2.2 Complex Power Concepts

The electrical power systems specialist is, in many instances, more concerned with electrical power in the circuit rather than current. As the power into an element is basically the product of the voltage across and current through it, it seems reasonable to swap the current for power without losing any information.

In treating sinusoidal steady-state behavior of an electric circuit, some further definitions are necessary. To illustrate, a cosine representation of the sinusoidal waveforms involved is used.

Consider an impedance element $\overline{Z} = Z/\phi$ connected to a sinusoidal voltage source $v(t)$ that is given by $v(t) = V_m \cos \omega t$. Figure 2.1 shows the typical load circuit. The instantaneous current in the circuit shown in Figure 2.1 is

$$i(t) = I_m \cos(\omega t - \phi),$$

where the current magnitude is

$$I_m = V_m/|Z|.$$

The instantaneous power is given by

$$p(t) = v(t)i(t) = V_m I_m[\cos(\omega t)\cos(\omega t - \phi)].$$

Using the trigonometric identity

$$\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)],$$

we can write the instantaneous power as

$$p(t) = \frac{V_m I_m}{2}[\cos\phi + \cos(2\omega t - \phi)].$$

The average power p_{av} is seen to be

$$p_{av} = \frac{V_m I_m}{2}\cos\phi. \tag{2.1}$$

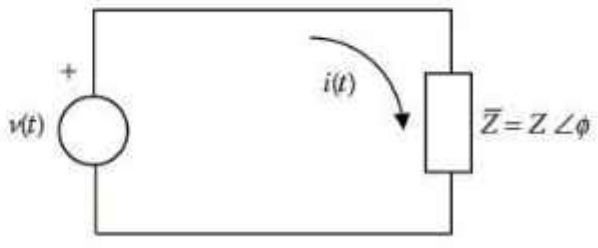


FIGURE 2.1
Load circuit.

Since through one cycle, the average of $\cos(2\omega t - \phi)$ is zero, this term therefore contributes nothing to the average of p.

It is more convenient to use the effective (or root mean square [rms]) values of voltage and current than the maximum values. Substituting $V_m = \sqrt{2}(V_{rms})$, and $I_m = \sqrt{2}(I_{rms})$, we get

$$p_{av} = V_{rms} I_{rms} \cos\phi. \tag{2.2}$$

Thus, the average power entering any network is the product of the effective values of terminal voltage and current and the cosine of the phase angle, which is called the power factor (PF). This applies to sinusoidal voltages and currents only. For a purely resistive load $\cos\phi = 1$, the current in the circuit is fully engaged in conveying power from the source to the load resistance. When reactive (inductive or capacitive) as well as resistive elements are present in the network, a component of the current in the circuit is engaged in conveying energy that is periodically stored in and discharged from the reactance. This stored energy, being shuttled to and from the magnetic field of an inductance or the electrostatic field of a capacitance, adds to the magnitude of the current in the circuit but does not add to the average power.

The average power in a circuit is called active power and the power that supplies the stored energy in reactive elements is informally called reactive power. Active power is denoted P, and the reactive power is designated Q. They are expressed as

$$P = VI\cos\phi \tag{2.3}$$

$$Q = VI\sin\phi, \tag{2.4}$$

where

V and I are the rms values of terminal voltage and current
ϕ is the phase angle by which the current lags the voltage
P and Q are of the same dimension (J/s)

We define a quantity called the complex or apparent power, designated S, of which P and Q are orthogonal components. By definition

$$\begin{aligned} S &= P + jQ = \overline{V}I^* \\ &= VI\cos\phi + jVI\sin\phi \\ &= VI(\cos\phi + j\sin\phi). \end{aligned}$$

Using Euler's identity, we thus have

$$\begin{aligned} S &= VIe^{j\phi} \\ &= VI\angle\phi. \end{aligned}$$

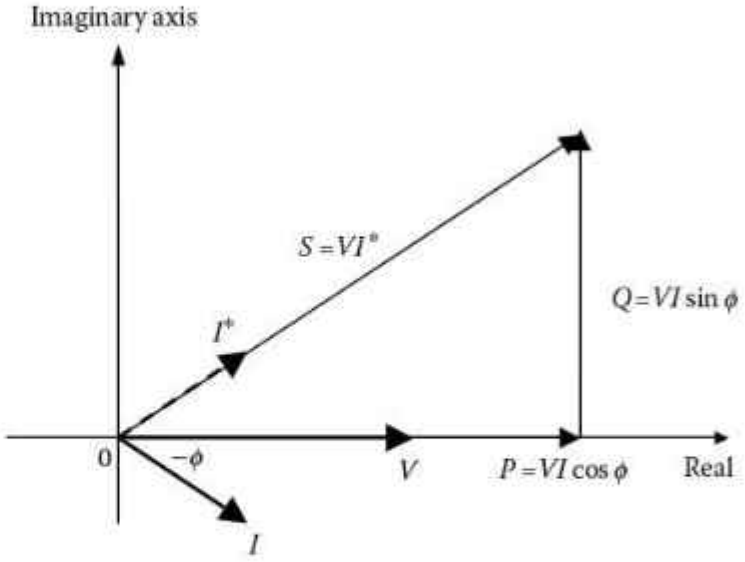


FIGURE 2.2
Phasor diagram for complex power relationships.

If we introduce the conjugate current defined by the symbol asterisk (*),

$$I^* = |I| \angle \phi,$$

it becomes obvious that an equivalent definition of complex or apparent power is

$$S = VI^*. \tag{2.5}$$

We can write the complex power in two alternative forms by using the relationships $\bar{V} = Z\bar{I}$ and $\bar{I} = Y\bar{V}$.

Multiplying the phasors by V, we obtain the complex power diagram in Figure 2.2. Inspection of the diagram as well as the previous development leads to a relation for the PF of the circuit:

$$\cos \phi = \frac{P}{|S|}.$$

2.3 Three-Phase Systems

The major portion of all the electric power presently used is generated, transmitted, and distributed using balanced three-phase voltage systems. The single-phase voltage sources referred to in Section 2.2 originate in many instances as part of a three-phase system [1]. Three-phase operation is preferable to single-phase because a three-phase winding makes more efficient use of generator copper and iron. Power flow in single-phase circuits was shown in the previous section to be pulsating. This drawback is not

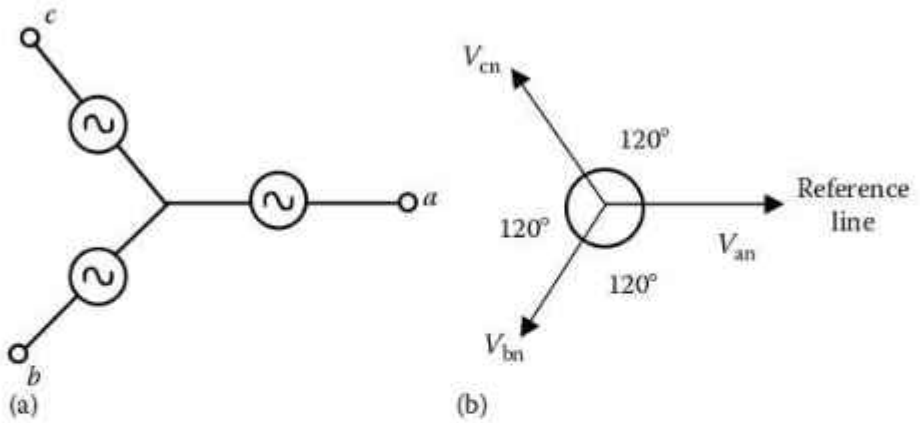


FIGURE 2.3
(a) Y-connected three-phase system and (b) corresponding phasor diagram.

present in a three-phase system as shown later. Also, three-phase motors start more easily and, having constant torque, run more satisfactorily than single-phase motors. However, the complications of additional phases are not compensated by the slight increase of operating efficiency when systems higher than three-phase systems are used.

A balanced three-phase voltage system consists of three single-phase voltages having the same magnitude and frequency but time-displaced from one another by 120°. Figure 2.3a shows a schematic representation where the single-phase voltage sources appear in a Y connection; a Δ configuration is also possible as discussed later. A phasor diagram showing each of the phase voltages is given in Figure 2.3b. As the phasors revolve at the angular frequency ω with respect to the reference line in the counterclockwise (designated as positive) direction, the positive maximum value first occurs for phase *a* and then in succession for phases *b* and *c*. Stated in a different way, to an observer in the phasor space, the voltage of phase *a* arrives first followed by that of *b* and then that of *c*. For this reason the three-phase voltage of Figure 2.3 is said to have the phase sequence *abc* (order, phase sequence, and rotation all mean the same thing). This is important for certain applications. For example, in three-phase induction motors, the phase sequence determines whether the motor rotates clockwise or counterclockwise.

2.3.1 Y-Connected Systems

With reference to Figure 2.4, the common terminal *n* is called the neutral or star (*Y*) point. The voltages appearing between any two of the line terminals *a*, *b*, and *c* have different relationships in magnitude and phase to the voltages appearing between any one line terminal and the neutral point *n*. The set of voltages V_{ab}, V_{bc}, and V_{ca} are the line voltages, and the set of voltages V_{an}, V_{bn}, and V_{cn} are referred to as the phase voltages. Consideration of the phasor diagrams provides the required relationships [1,3].

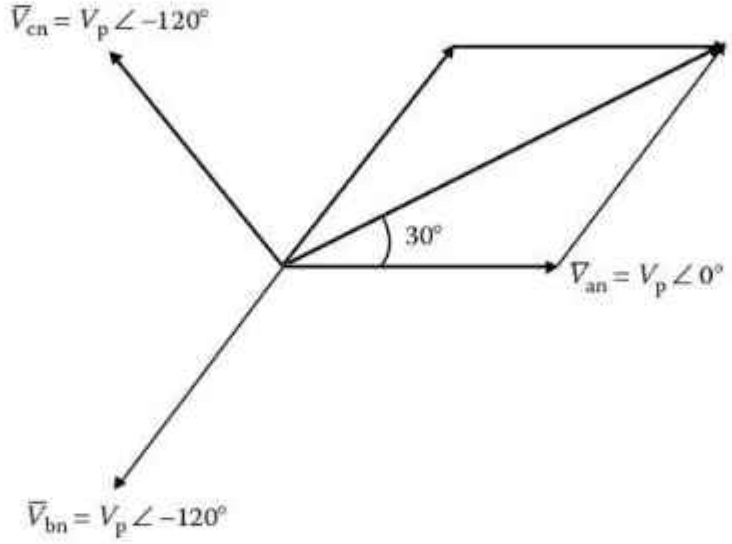


FIGURE 2.4
Phase and magnitude relations between the phase and line voltage of a Y-connection.

The effective values of the phase voltages are shown in Figure 2.5 as V_{an}, V_{bn}, and V_{cn}. Each has the same magnitude, and each is displaced 120° from the other two phasors. To obtain the magnitude and phase angle of the line voltage from a to b (i.e., V_{ab}), we apply Kirchhoff's voltage law:

$$V_{ab} = V_{an} + V_{nb}. \tag{2.6}$$

Equation 2.6 states that the voltage existing from a to b is equal to the voltage from a to n (i.e., V_{an}) plus the voltage from n to b. Thus, Equation 2.6 can be rewritten as

$$V_{ab} = V_{an} - V_{bn}. \tag{2.7}$$

Since for a balanced system, each phase voltage has the same magnitude, let us set

$$|V_{an}| = |V_{bn}| = |V_{cn}| = V_p, \tag{2.8}$$

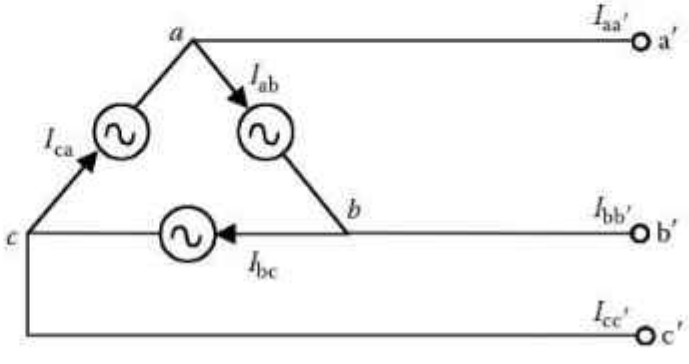


FIGURE 2.5
Relation between phase and line currents in a Δ connection.

where V_p denotes the effective magnitude of the phase voltage. Accordingly, we may write

$$V_{an} = V_p \angle 0^\circ \tag{2.9}$$

$$V_{bn} = V_p \angle -120^\circ \tag{2.10}$$

$$V_{cn} = V_p \angle -240^\circ = V_p \angle 120^\circ. \tag{2.11}$$

Substituting Equations 2.9 and 2.10 in Equation 2.7 yields

$$\begin{aligned} V_{ab} &= V_p(1 - 1\angle -120^\circ) \\ &= \sqrt{3}V_p \angle 30^\circ. \end{aligned} \tag{2.12}$$

Similarly we obtain

$$V_{bc} = \sqrt{3}V_p \angle -90^\circ \tag{2.13}$$

$$V_{ca} = \sqrt{3}V_p \angle 150^\circ. \tag{2.14}$$

The expressions obtained above for the line voltages, V_L, show that they constitute a balanced three-phase voltage system whose magnitudes are $\sqrt{3}$ times as those of the phase voltages. Thus, we write

$$V_L = \sqrt{3}V_p. \tag{2.15}$$

A current flowing out of a line terminal a (or b or c) is the same as that flowing through the phase source voltage appearing between terminals n and a (or n and b or n and c). We can thus conclude that for a Y-connected three-phase source, the line current equals the phase current. Thus

$$I_L = I_p, \tag{2.16}$$

where

I_L denotes the effective value of the line current
I_p denotes the effective value for the phase current

2.3.2 Delta-Connected Systems

We now consider the case when the three single-phase sources are rearranged to form a three-phase Δ connection as shown in Figure 2.6. It is clear from inspection of the circuit shown that the line and phase voltages have the same magnitude:

$$|V_L| = |V_p|. \tag{2.17}$$

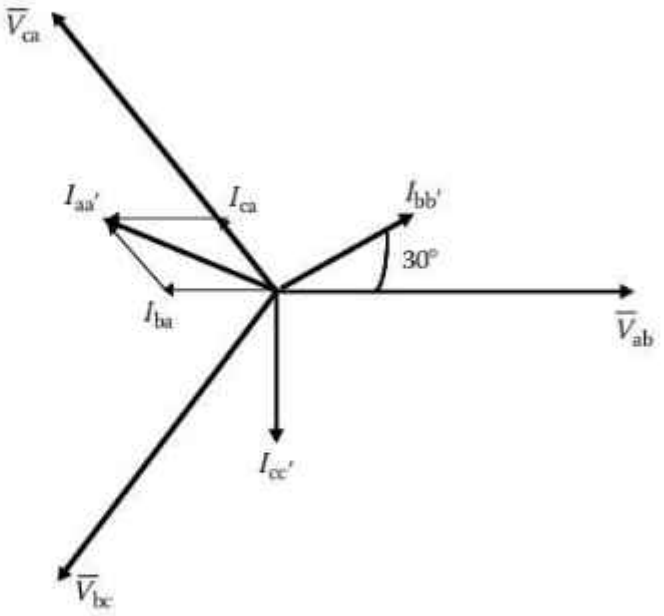


FIGURE 2.6
A Δ-connected three-phase source.

The phase and line currents, however, are not identical, and the relationship between them can be obtained using Kirchhoff's current law at one of the line terminals.

In a manner similar to that adopted for the Y-connected source, let us consider the phasor diagram shown in Figure 2.6. Assume the phase currents to be

$$I_{ab} = I_p \angle 0$$
$$I_{bc} = I_p \angle -120^\circ$$
$$I_{ca} = I_p \angle 120^\circ.$$

The current that flows in the line joining a to a' is denoted $I_{aa'}$ and is given by

$$I_{aa'} = I_{ca} - I_{ab}.$$

As a result, we have

$$I_{aa'} = I_p[1\angle 120^\circ - 1\angle 0],$$

which simplifies to

$$I_{aa'} = \sqrt{3} I_p \angle 150^\circ.$$

Similarly

$$I_{bb'} = \sqrt{3} I_p \angle 30^\circ$$
$$I_{cc'} = \sqrt{3} I_p \angle -90^\circ.$$

2.3.3 Power Relationships

Note that a set of balanced three-phase currents yields a corresponding set of balanced line currents whose magnitudes are $\sqrt{3}$ times the magnitudes of the phase values:

$$I_L = \sqrt{3} I_p,$$

where I_L denotes the magnitude of any of the three line currents.

Assume that a three-phase generator is supplying a balanced load with the three sinusoidal phase voltages:

$$\begin{aligned} v_a(t) &= \sqrt{2} V_p \sin \omega t \\ v_b(t) &= \sqrt{2} V_p \sin(\omega t - 120^\circ) \\ v_c(t) &= \sqrt{2} V_p \sin(\omega t + 120^\circ), \end{aligned}$$

with the currents given by

$$\begin{aligned} i_a(t) &= \sqrt{2} I_p \sin(\omega t - \phi) \\ i_b(t) &= \sqrt{2} I_p \sin(\omega t - 120^\circ - \phi) \\ i_c(t) &= \sqrt{2} I_p \sin(\omega t + 120^\circ - \phi), \end{aligned}$$

where ϕ is the phase angle between the current and voltage in each phase.

The total power in the load is

$$p_{3\phi}(t) = v_a(t) i_a(t) + v_b(t) i_b(t) + v_c(t) i_c(t).$$

This can be expanded as

$$\begin{aligned} p_{3\phi}(t) = 2 V_p I_p [&\sin(\omega t) \sin(\omega t - \phi) \\ &+ \sin(\omega t - 120^\circ) \sin(\omega t - 120^\circ - \phi) \\ &+ \sin(\omega t + 120^\circ) \sin(\omega t + 120^\circ - \phi)]. \end{aligned}$$

Using a trigonometric identity, we get

$$p_{3\phi}(t) = V_p I_p \{3 \cos\phi - [\cos(2\omega t - \phi) + \cos(2\omega t - 240^\circ - \phi) + \cos(2\omega t + 240^\circ - \phi)]\}.$$

Note that the last three terms in the above equation add up to zero. Thus, we obtain

$$p_{3\phi}(t) = 3 V_p I_p \cos\phi.$$

When referring to the voltage level of a three-phase system, by convention, one invariably understands the line voltages. From the above discussion, the relationship between the line and phase voltages in a Y-connected system is

$$|V_L| = \sqrt{3}|V|.$$

The power equation thus reads in terms of line quantities:

$$p_{3\phi} = \sqrt{3}|V_L||I_L|\cos\phi.$$

We note that the total instantaneous power is constant, having a magnitude of three times the real power per phase.

We may be tempted to assume that the reactive power is of no importance in a three-phase system since the Q terms cancel out. However, this situation is analogous to the summation of balanced three-phase currents and voltages that also cancel out. Although the sum cancels out, these quantities are still very much in evidence within each phase.

We extend the concept of complex or apparent power (S) to three-phase systems by defining

$$S_{3\phi} = 3V_p I_p^*,$$

where the active and reactive powers are obtained from

$$S_{3\phi} = P_{3\phi} + jQ_{3\phi}$$

as

$$P_{3\phi} = 3|V_p||I_p|\cos\phi$$
$$Q_{3\phi} = 3|V_p||I_p|\sin\phi.$$

In terms of line values, we can assert that

$$S_{3\phi} = \sqrt{3}V_L I_L^*$$

and

$$P_{3\phi} = \sqrt{3}|V_L||I_L|\cos\phi$$
$$Q_{3\phi} = \sqrt{3}|V_L||I_L|\sin\phi.$$

2.4 Per Unit Representation

In power system analysis, it is usually convenient to use a per unit system to normalize system variables. The per unit system offers computational simplicity by eliminating units and expressing system quantities as a dimensionless ratio. Thus

$$\overline{X}_{(\text{in per unit})} = \frac{X_{(\text{actual quantity})}}{X_{(\text{base value of equation})}}.$$

A well-chosen per unit system can minimize computational effort, simplify evaluation, and facilitate understanding of system characteristics. Some base quantities may be chosen independently and quite arbitrarily, while others follow automatically depending on fundamental relationships between system variables. Normally, the base values are chosen so that the principal variables will be equal to one per unit under rated operating conditions.

In the case of a synchronous machine, the per unit system may be used to remove arbitrary constants and simplify mathematical equations so that they may be expressed in terms of equivalent circuits. The basis for selecting the per unit system for the stator is straightforward, but it requires careful consideration for the rotor. The L_{ad}-base reciprocal per unit system is discussed here.

The following base quantities for the stator are chosen (denoted by subscripts).

$e_{s\,base}$, peak value of rated line-to-line voltage (V)
$i_{s\,base}$, peak value of rated line-to-line current (A)
f_{base}, rated frequency (Hz)

The base value of each of the remaining quantities is automatically set and depends on the above as follows:

$$\omega_{base} = 2\pi f_{base} \text{ (electrical rad/s)}$$

$$\omega_{m\,base} = \omega_{base}\frac{2}{\pi f} \text{ (mechanical rad/s)}$$

$$Z_{s\,base} = \frac{e_{s\,base}}{i_{s\,base}} \ (\Omega)$$

$$L_{s\,base} = \frac{e_{s\,base}}{\omega_{base}} \ (\text{H})$$

$$\psi_{s\,base} = L_{s\,base} \times i_{s\,base} = \frac{e_{s\,base}}{\omega_{m\,base}} \text{ (Weber turns)}$$

$$VA_{3\phi,base} = 3E_{rms\,base}I_{rms\,base} = \frac{3}{2}e_{s\,base} \times i_{s\,base} \text{ (VA)}.$$

2.5 Synchronous Machine Modeling

In power system stability analysis, there are several types of models used for representing the dynamic behavior of the synchronous machine. These models are deduced by using some approximations to the basic machine equations. Section 2.5 begins with a brief introduction to machine equations [2,5].

The stator voltage equations expressed in per unit notations are given by

$$\begin{aligned}
\bar{e}_d &= \frac{d\bar{\psi}_d}{d\bar{t}} - \bar{\psi}_q\bar{\omega} - \bar{R}_a\bar{i}_d \\
\bar{e}_q &= \frac{d\bar{\psi}_q}{d\bar{t}} + \bar{\psi}_d\bar{\omega} - \bar{R}_a\bar{i}_q \\
\bar{e}_0 &= \frac{d\psi_0}{d\bar{t}} - \bar{R}_a\bar{i}_0.
\end{aligned} \tag{2.18}$$

The corresponding flux linkage equations may be written as

$$\begin{aligned}
\bar{\psi}_d &= -\bar{L}_d\bar{i}_d + \bar{L}_{afd}\bar{i}_{fd} + \bar{L}_{akd}\bar{i}_{kd} \\
\bar{\psi}_q &= -\bar{L}_q\bar{i}_q + \bar{L}_{akq}\bar{i}_q \\
\bar{\psi}_0 &= -\bar{L}_0\bar{i}_0.
\end{aligned} \tag{2.19}$$

The rotor circuit base quantities should be chosen so as to make the flux linkage equations simple by satisfying the following:

1. Per unit mutual inductances between different windings are to be reciprocal. This will allow the synchronous machine model to be represented by simple equivalent circuits.
2. All per unit mutual inductances between stator and rotor circuits in each axis are to be equal.

The following base quantities for the rotor are chosen, in view of the L_{ad}-base per unit system choice.

$$\begin{aligned}
\bar{e}_d &= \frac{d\bar{\psi}_d}{d\bar{t}} - \bar{\psi}_q\bar{\omega} - \bar{R}_a\bar{i}_d \\
\bar{e}_q &= \frac{d\bar{\psi}_q}{d\bar{t}} + \bar{\psi}_d\bar{\omega} - \bar{R}_a\bar{i}_q \\
\bar{e}_0 &= \frac{d\bar{\psi}_0}{d\bar{t}} - \bar{R}_a\bar{i}_0.
\end{aligned} \tag{2.20}$$

Per unit rotor flux linkage equations are given by

$$\begin{aligned}\bar{\psi}_d &= -\bar{L}_d\bar{i}_d + \bar{L}_{afd}\bar{i}_{fd} + \bar{L}_{akd}\bar{i}_{kd}\\ \bar{\psi}_q &= -\bar{L}_q\bar{i}_q + \bar{L}_{akq}\bar{i}_q\\ \bar{\psi}_0 &= -\bar{L}_0\bar{i}_0.\end{aligned} \tag{2.21}$$

Since all quantities in Equations 2.18 through 2.21 are in per unit, the super bar notation is dropped in subsequent discussions.

If the frequency of the stator quantities is equal to the base frequency, the per unit reactance of a winding is numerically equal to the per unit inductance. For example:

$$X_d = 2\pi f L_d \ (\Omega).$$

Dividing by $Z_{s\,base} = 2\pi f_{base} L_{s\,base}$, if $f = f_{base}$, the per unit values of X_d and L_d are equal.

2.5.1 Classical Representation of the Synchronous Machine

The per unit equations completely describe the electrical and dynamic performance of a synchronous machine. However, except for the analysis of very small systems, these equations cannot be used directly for system stability studies. Some simplifications and approximations are required for the representation of synchronous machines in stability studies. For stability analysis of large systems, it is necessary to neglect the transformer voltage terms $p\psi_d$ and $p\psi_q$ and the effect of speed variations. Therefore, the machine equation [3] described by Equations 2.20 and 2.21 becomes

$$\begin{cases} e_d = -\psi_q - R_a i_a \\ e_q = \psi_d - R_a i_a \\ e_{fd} = p\psi_{fd} + R_{fd} i_{fd}. \end{cases} \tag{2.22}$$

For studies in which the period of analysis is small in comparison to T'_{d0} the machine model is often simplified by assuming E'_q constant throughout the study period. This assumption eliminates the only differential equation associated with the electrical characteristics of the machine. A further approximation, which simplifies the machine representation significantly, is to ignore transient saliency by assuming that $x'_d = x'_q$ and to assume that the flux linkage also remains constant. With these assumptions, the voltage behind the transient impedance $R_a + jx'_d$ has a constant magnitude. The equivalent circuit is shown in Figure 2.7. The machine terminal voltage phasor is represented by

$$\tilde{V}_t = E'\angle\delta - (R_a + jx'_d)\tilde{I}_t.$$

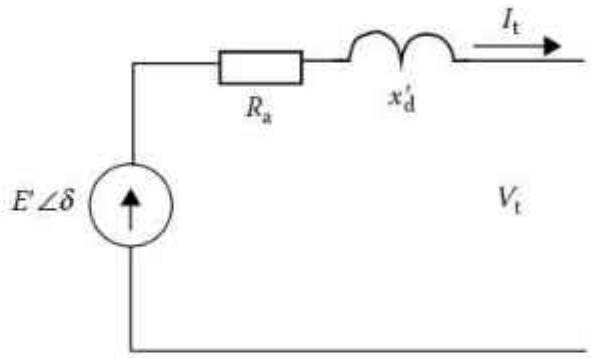


FIGURE 2.7
Equivalent circuit of machine.

2.6 Reactive Capability Limits

Synchronous generators are usually rated in terms of the maximum kVA load at a specific voltage and PF (often 80%, 85%, or 90% lagging) that they can carry continuously without overheating. The active power output of the generator is usually limited to a value within the kVA rating by the capability [4] of its prime mover. By virtue of its voltage regulating system, the machine normally operates at a constant voltage whose value is within +5% of rated voltage. When the active-power loading and voltage are fixed, either armature or field heating limits the allowable reactive-power loading. A typical set of reactive-power capability curves for a large turbine generator are shown in Figure 2.8. They give the maximum reactive-power loadings corresponding

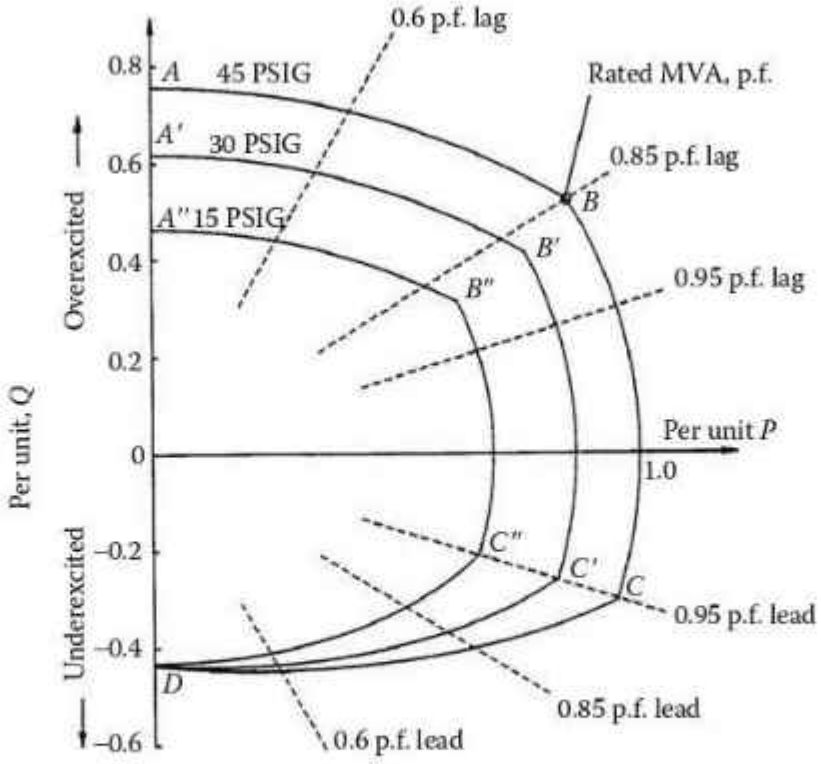


FIGURE 2.8
Reactive capability curves of a hydrogen-cooled generator at rated voltage.

to various power loadings with operation at rated voltage. Armature heating is the limiting factor in the region from unity to the rated PF (0.85 in Figure 2.8). For lower PFs, field heating is limiting. Such a set of curves forms a valuable guide in planning and operating the system of which the generator is a part.

Also shown in Figure 2.8 is the effect of increased hydrogen pressure on allowable machine loadings. The PF at which a synchronous motor operates, and hence its armature current, can be controlled by adjusting its field excitation. The curve showing the relation between armature current and field current at a constant terminal voltage and with a constant shaft load is known as a V-curve because of its characteristic shape.

2.7 Prime Movers and Governing Systems

The prime sources of electrical energy supplied by utilities are the kinetic energy of water and the thermal energy derived from fossil fuels and nuclear fission. The prime movers convert these sources of energy into mechanical energy that, in turn, is converted to electrical form by the synchronous generator. The prime mover governing system provides a means of controlling power and frequency. The functional relationships among the basic elements associated with power generation and control are shown in Figure 2.9. This section introduces the models for hydraulic and steam turbines and their respective governing systems.

2.7.1 Hydraulic Turbines and Governing Models

The hydraulic turbine model describes the characteristics of gate opening m and output mechanical power. In power system dynamic analysis, the

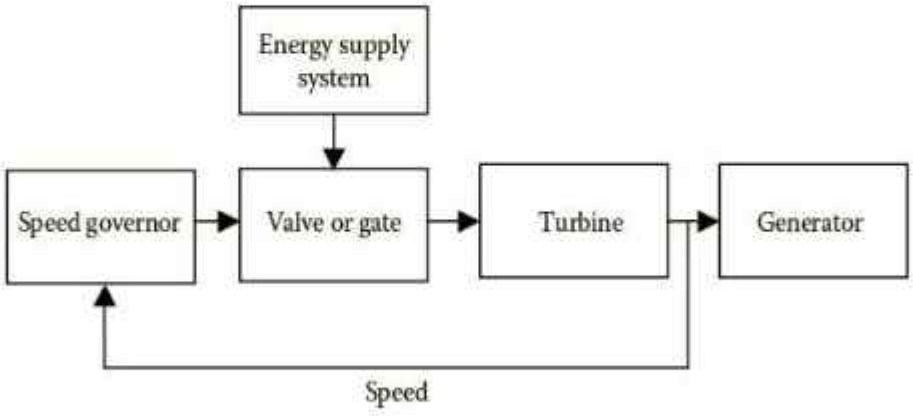


FIGURE 2.9
Power generation and control.

hydraulic turbine is usually modeled by an ideal lossless turbine along with consideration of the "water hammer" effect caused by the water inertia, given by

$$\frac{P_m}{\mu} = \frac{1 - T_w s}{1 + T_w s}, \tag{2.23}$$

where T_w is water-starting time.

Because of the water hammer effect, a change in gate position produces an initial turbine power change that is opposite to that which is sought. For stable control performance, a large transient (temporary) droop with a long resetting time is required. This is accomplished by introducing a transient gain-reduction compensation in the governing system. The compensation retards or limits the gate movement until the water flow and power output have time to catch up. The governing system model is shown in Figure 2.10.

The governing system model is given by

$$\begin{aligned} \frac{dX_1}{dt} &= \eta \\ T_G \frac{d\mu}{dt} &= -\mu + X_1 \\ T_R \frac{dX_2}{dt} &= -X_2 + R_T T_R \eta \\ T_P \frac{d\eta}{dt} &= -\eta + K_s(\omega_{ref} - \omega_r - R_p X_1 - X_2), \end{aligned} \tag{2.24}$$

where

T_p is the pilot valve and servomotor time constant
K_s is the servo gain
T_a is the main servo time constant

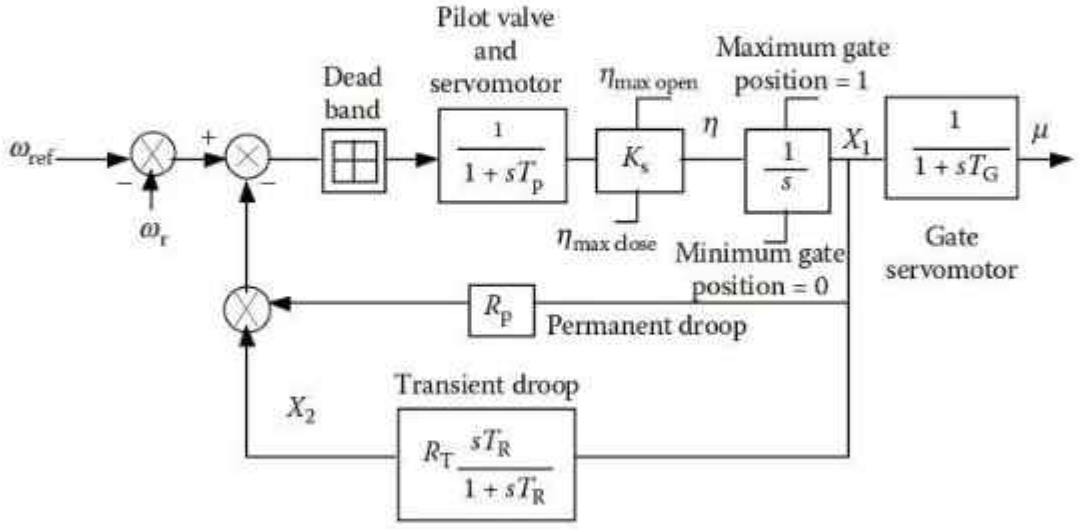


FIGURE 2.10
Governing system model.

R_P is the permanent droop
R_T is the temporary droop
T_R is the reset time
$\eta_{\max\ \text{open}}$ is the maximum gate opening rate
$\eta_{\max\ \text{close}}$ is the maximum gate closing rate
μ is the gate position

2.7.2 Steam Turbines and Governing System Models [2,5]

A steam turbine converts stored energy of high pressure and high temperature steam into rotating energy. The input of the steam turbine is the control valve position (ΔV_{cv}), while its output is the torque (ΔT_m). In power stability analysis, a first-order model is used for a steam turbine; that is

$$\frac{\Delta T_m}{\Delta V_{cv}} = \frac{1}{1 + sT_{CH}}, \tag{2.25}$$

where T_{CH} is the time constant.

Comparing the turbine models for hydraulic and steam turbines, it is clear that the response of a steam turbine has no peculiarity such as that exhibited by a hydraulic turbine due to water inertia. The governing requirements of steam turbines, in this respect, are more straightforward. There is no need for transient droop compensation.

The governing system model is given by

$$\begin{aligned} T_G \frac{d\Delta V_{cv}}{dt} &= -\Delta V_{cv} + X_1 \\ \frac{dX_1}{dt} &= K_s(\omega_{ref} - \omega_r - R_P X_1). \end{aligned} \tag{2.26}$$

A typical governing model for a steam turbine is shown in Figure 2.11.

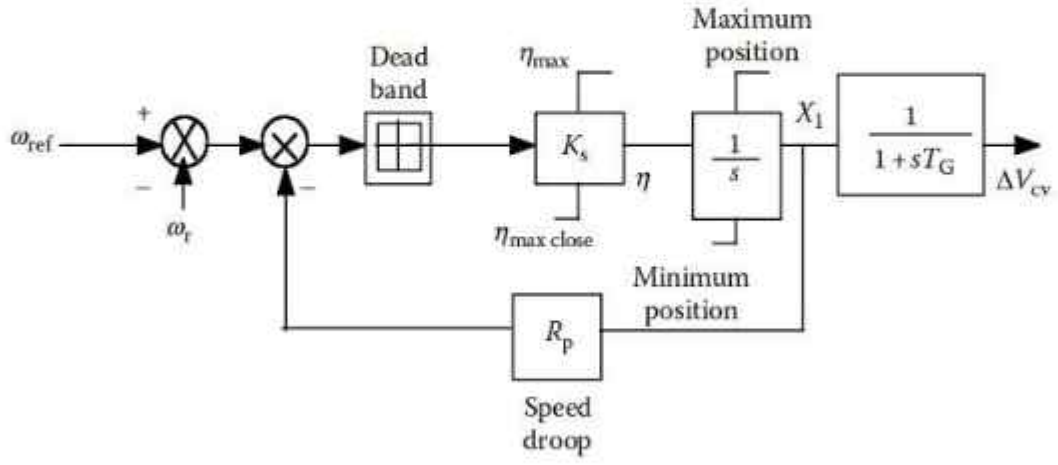


FIGURE 2.11
Governing model for steam turbine.

2.8 Automatic Gain Control

Within an energy management system, the generation scheduling and control function control the electrical power output of generators so as to supply the continuously changing customer power demand in an economical manner. This function is largely provided by an automatic generation control (AGC) program operating within the control center computer. The power system dispatcher also plays an important role, interacting with the program to incorporate current operating conditions. The basic objectives of power system operation during normal operating conditions associated with AGC are

1. Matching total system generation to total system load
2. Regulating system electrical frequency error to zero
3. Distributing system generation among control areas so that net area tie flows match net area tie flow schedules
4. Distributing area generation among area generation sources so that area operating costs are minimized

The first objective is conventionally associated with the terms primary or governor speed control; turbine speed governors respond proportionally to local frequency deviations and normally bring the frequency rate of change to zero within a time frame of several seconds. The latter three objectives are accomplished by supplementary controls directed from area control centers. The second and third AGC objectives are classically associated with the regulation function, or load-frequency control, while the fourth is associated with the economic dispatch function of AGC. The regulation and economic dispatch functions typically operate in time frames of several seconds and several minutes, respectively.

2.8.1 Power Control in a Multigenerator Environment

Consider the case of a number of generating units connected by a transmission system, with local loads at each generator bus as shown in Figure 2.12. We are interested in active power flows and assume that we can use the active power model with all voltages set equal to their nominal values. The objective of the analysis is to obtain an incremental model relating power commands, electrical and mechanical power outputs, and power angles for the different generators. We note that the analysis may be easily extended to include pure load buses in the model; however, here we assume that every bus is a generator bus.

At each bus, $P_{gi} = P_{Di} + P_i$, where P_i is the total injected active power from the ith bus into the transmission system. Assume an operating point

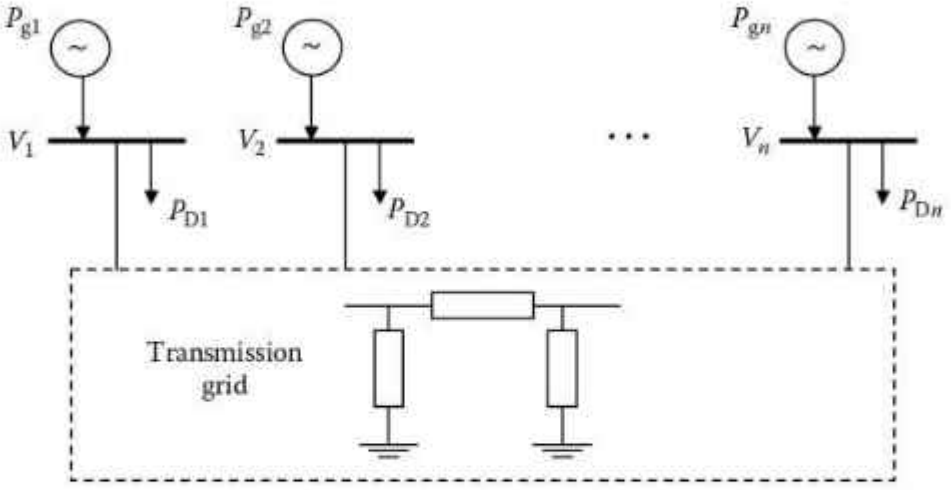


FIGURE 2.12
Simplified network model for an n-generator power system.

given by $P_{gi}^{o} = P_{Di}^{o} + P_i^{o}$ and then $P_{gi}^{o} = P_{gi}^{o} + \Delta P_{gi}$, $P_{Di}^{o} = P_{Di}^{o} + \Delta P_{Di}$, and $P_i^{o} = P_i^{o} + \Delta P_i$; we get the relationship between increments:

$$\Delta P_{gi} = \Delta P_{Di}^{o} + \Delta P_i.$$

For each generator we use the linearized version of the swing equation. Thus, we consider

$$M_i \Delta\ddot{\delta}_i + D_i \Delta\dot{\delta}_i + \Delta P_{Gi} = \Delta P_{Mi}, \tag{2.27}$$

where δ_i is the angle of the internal voltage of the ith generator. We number the generators so that the ith generator is connected to the ith bus.

We next consider the behavior of P_{Di}. In general, P_{Di} depends on both the voltage magnitude and frequency. In our active power model we assume that the voltage magnitudes are fixed at their nominal values, and we therefore need not consider the voltage magnitude dependence. Concerning the frequency dependence, we assume that P_{Di} increases with frequency. This assumption is in accordance with observed behavior. We would also like to be able to introduce changes in load as external (disturbance) inputs. Thus, we can imagine the sudden increments in load due to the switching of an electrical device that draws a specified power. In accordance with this description, for small changes in frequency, we assume that

$$P_{Dei} = P_{Di}^{o} + \frac{\partial P_{Di}(\omega^{o})}{\partial \omega_i} \Delta\omega_i + \Delta P_{Li}, \tag{2.28}$$

where the first two terms describe the linearization of the frequency dependence of the load and ΔP_{Li} is the change of load input, by switching, referred to previously. Note that the operating-point frequency is ω^{o}, which is

expected to be close to but not necessarily equal to the nominal system frequency $\omega_o = 2\pi 60$. In defining the frequency at each bus, we note that the instantaneous (phase a) voltage is

$$v_i(t) = \sqrt{2}|V_i| \cos\left[\omega^o t + \theta_i^o + \Delta\theta_i(t)\right]. \tag{2.29}$$

Thus, by taking the time derivative of the argument of the cosine function, we get

$$\omega_i = \omega^o + \Delta\dot{\theta}_i \tag{2.30}$$

and thus

$$\Delta\omega_i = \omega_i - \omega^o = \Delta\dot{\theta}_i. \tag{2.31}$$

Defining $D_{Li} = \partial P_{Di}(\omega_o)/\partial\omega_i$, and using Equation 2.31, we can replace Equation 2.28 with

$$\Delta P_{Di} = D_{Li}\Delta\dot{\theta}_i + \Delta P_{Li}. \tag{2.32}$$

Also, we can get

$$M_i\Delta\ddot{\delta}_i + D_i\Delta\dot{\delta}_i + D_{Li}\Delta\dot{\theta}_i + \Delta P_{Li} + \Delta P_i = \Delta P_{Mi}. \tag{2.33}$$

We next calculate ΔP_i, assuming that the line admittance parameters are purely imaginary,

$$P_i = \sum_{k=1}^{n} |V_i||V_k|B_{ik}\sin(\theta_i - \theta_j), \tag{2.34}$$

where θ_i is the phase angle of the bus voltage V_i. Assuming that voltage magnitudes $|V_i|$ are constants, then the terms $|V_i||V_k|B_{ik}$ are constants by linearizing Equation 2.34 around the operating point. For variety, instead of using the Taylor series, we can proceed as follows, using trigonometric identities.

$$\begin{aligned} P_i = P_i^o + \Delta P_i &= \sum_{k=1}^{n} |V_i||V_j|B_{ik}\sin\left(\theta_i^o + \Delta\theta_i - \theta_k^o - \Delta\theta_k\right) \\ &= \sum_{k=1}^{n} |V_i||V_j|B_{ik}\sin\left(\theta_i^o - \theta_k^o\right)\cos(\Delta\theta_i - \Delta\theta_k) \\ &\quad + \cos\left(\theta_i^o - \theta_k^o\right)\sin(\Delta\theta_i - \Delta\theta_k). \end{aligned} \tag{2.35}$$

This result is exact. As we let the increments go to zero we get

$$\Delta P_i = \sum_{k=1}^{n} |V_i||V_j|B_{ik}\cos\left(\theta_i^o - \theta_k^o\right)(\Delta\theta_i - \Delta\theta_k), \tag{2.36}$$

which is the same as the linearization result found by using the Taylor series.

Defining $T_{ik} = |V_i||V_k|B_{ik}\cos\left(\theta_i^o - \theta_k^o\right)$, we get

$$\Delta P_i = \sum_{k=1}^{n} T_{ik}(\Delta\theta_i - \Delta\theta_k). \tag{2.37}$$

The constants T_{ik} are called stiffness or synchronizing power coefficients. The larger the T_{ik}, the greater the exchange of power for a given change in bus phase angle. We are now in a position to substitute Equation 2.37 into Equation 2.33. First, however, we make a further approximation. We assume coherency between the internal and the terminal voltage phase angles of each generator so that these angles tend to "swing together." Stated differently, we assume that the increments $\Delta\delta_i$ and $\Delta\theta_i$ are equal. The assumption that $\Delta\delta_i = \Delta\theta_i$ significantly simplifies the analysis and gives results that are qualitatively correct.

Making the assumption, and substituting Equation 2.37 into Equation 2.33, we get for i 2, . . . , n,

$$M_i\Delta\ddot{\delta}_i + \widetilde{D}_i\Delta\dot{\delta}_i + \Delta P_i = \Delta P_{Mi} - \Delta P_{Li}, \tag{2.38}$$

where

$\widetilde{D}_i = D_i + D_{Li}$

$\Delta P_i = \sum_{k=1}^{n} T_{ik}(\Delta\delta_i - \Delta\delta_k)$

From the way D_{Li} adds to D_i, we expect the (positive) load dependence on frequency to contribute to system damping.

The preceding relations are shown in the form of block diagram in Figure 11.10. The reader is invited to check that with $K_{Pi} = 1/\widetilde{D}_i$ and $T_{Pi} = M_i/\widetilde{D}_i$, Figure 11.10 represents Equation 2.38 in form of block diagram. In Figure 2.13

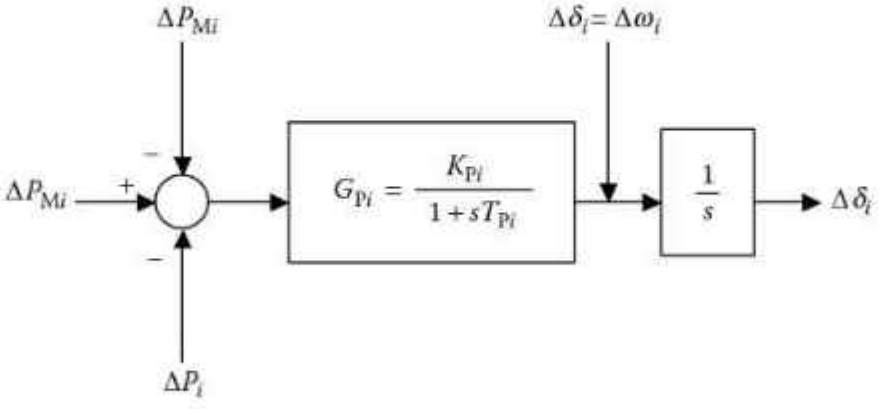


FIGURE 2.13
Generator block diagram.

we have $\Delta\omega_i$ as an output and ΔP_{Mi} as an input and can close the power control loop by introducing the turbine governor block diagram.

2.8.2 AGC System Models

2.8.2.1 Case A: Two Generating Units

In this case, we have two synchronizing coefficients to consider, T_{12} and T_{21}.

$$T_{12} = |V_1||V_2|B_{12}\cos\left(\theta_1^o - \theta_2^o\right)$$
$$T_{21} = |V_1||V_2|B_{21}\cos\left(\theta_2^o - \theta_1^o\right).$$

Since $T_{12} = T_{21}$, the block diagram for the two units may be combined and we can describe the operation of the system in qualitative terms as follows. Starting in the steady-state $\Delta\omega_1 = \Delta\omega_2 = 0$, suppose that additional load is switched onto bus 2 (i.e., ΔP_{L2} is positive). Initially, the mechanical power input to generator 2 does not change; the power imbalance is supplied from the rotating kinetic energy. Thus ω_2 begins to drop (i.e., $\Delta\omega_2$ goes negative). As a consequence, $\Delta\delta_2$ decreases and the phase angle $\delta_1 - \delta_2$ across the connecting transmission line increases. Thus the transferred power P_{12} increases. The sending-end power P_{12} and the receiving-end power $-P_{12}$ both increase. Unit 1 sees an increased power demand and ω_1 begins to drop. Each governor senses a frequency (speed) drop and acts to increase the mechanical power output from its turbine. In the steady-state there is a new (lower) system frequency and an increased P_{12}. The frequency may then be restored and P_{12} adjusted, if desired, by operator action (i.e., by adjustment of P_{c1} and P_{c2}).

We next consider an example in which we calculate the steady-state change frequency by automatic (governor) control and see the benefit of the interconnection in quantitative terms. We consider the more general case when both P_{L1} and P_{L2} change.

1. *Controlled two-area systems*: The response curves indicate that, as in the single-area case, we must add integral control to our system. Let us state, first, the minimum requirements the system should meet.
2. *Suggested control-system specifications*: We require that our system meet the four-point specifications that we stipulated for the single-area system. In addition, we require that the steady-state tie line power variation, following a step load change, must be zero. This requirement guarantees that each area, in steady state, absorbs its own load, the guiding principle in pool operation.
3. *Tie line bias control strategy*: Since we must now use a strategy that will cause both the frequency and the tie line deviations to vanish, we adopt integral control, as in the single-area case, but with the tie line deviation added to our area control error; that is, we attempt

$$\begin{aligned} \mathrm{ACE}_1 &= \Delta P_{\mathrm{tie1}} + B_1 \Delta f_1 \\ \mathrm{ACE}_2 &= \Delta P_{\mathrm{tie2}} + B_2 \Delta f_2. \end{aligned} \tag{2.39}$$

The speed changer commands are thus of the form

$$\begin{aligned} \Delta P_{c1} &\triangleq -K_{I1} \int (\Delta P_{\mathrm{tie1}} + B_1 \Delta f_1)\, \mathrm{d}t \\ \Delta P_{c2} &\triangleq -K_{I2} \int (\Delta P_{\mathrm{tie2}} + B_2 \Delta f_2)\, \mathrm{d}t. \end{aligned} \tag{2.40}$$

The constants K_{I1} and K_{I2} are integrator gains, and the constants B_1 and B_2 are the frequency bias parameters. The minus signs must be included since each area should increase its generation if either its frequency error Δf_i or its tie line power increment $\Delta P_{\mathrm{tie},i}$ is negative.

4. *Static system response*: The chosen strategy will eliminate the steady-state frequency and tie line deviations for this reason: following a step load change in either area, a new static equilibrium, if it exists, can be achieved only after the speed changer commands ΔP_{c1} and ΔP_{c2} have reached constant values. But this evidently requires that both integrands in Equation 2.40 be zero; that is

$$\begin{aligned} \Delta P_{\mathrm{tie1,stat}} + B_1 \Delta f_{\mathrm{stat}} &= 0 \\ \Delta P_{\mathrm{tie2,stat}} + B_2 \Delta f_{\mathrm{stat}} &= 0. \end{aligned} \tag{2.41}$$

In view of Equation 2.41, these conditions can be met only if

$$\Delta f_{\mathrm{stat}} = \Delta P_{\mathrm{tie1,stat}} = \Delta P_{\mathrm{tie2,stat}} = 0.$$

Note that this result is independent of the B_1 and B_2 values. In fact, one of the bias parameters (but not both) can be zero, and we still have a guarantee that the previous equation is satisfied.

5. *Prime mover response*: At this point, we can construct a block diagram of a governor-prime mover obtaining mass/load model. Suppose that this governor experiences a step increase in load

$$\Delta P_{\mathrm{L}}(s) = \frac{\Delta P_{\mathrm{L}}}{s}. \tag{2.42}$$

The transfer function relating the load change ΔP_{L} to the frequency change $\Delta\omega$ is

$$\Delta\omega(s) = \Delta P_{\mathrm{L}}(s) \left[\frac{(-1/Ms + D)}{1 + (1/R)(1/1 + sT_{\mathrm{G}})(1/1 + sT_{\mathrm{CH}})(1/Ms + D)} \right]. \tag{2.43}$$

The steady-state value of $\Delta\omega(s)$ may be found by

$$\Delta\omega_{\text{steady state}} = \lim_{s\to 0}[s\Delta\omega(s)] = \frac{-\Delta P_L(1/D)}{1+(1/R)(1/D)} = \frac{-\Delta P_L}{(1/R)+D}. \tag{2.44}$$

Note that if D were zero, the change in speed would simply be $\Delta\omega = -R\Delta P_L$. If several generators (each having its own governor and prime mover) were connected to the system, the frequency change would be

$$\Delta\omega = \frac{-\Delta P_L}{(1/R_1)+(1/R_2)+\cdots+(1/R_n)+D}. \tag{2.45}$$

2.9 Transmission Subsystems [3,6]

The transmission network subsystem involves complexities in busbar interties, transformers, and in the transmission links themselves.

1. Busbars are often interconnected by circuit breakers in such a manner to allow switching flexibility (i.e., single or groups of circuits may be outaged without undesirable outaging of key buses or other circuits). A bus tie circuit breaker presents somewhat of a mathematical difficulty in that opening of the breaker in the computer study requires "outaging" of a zero impedance tie. The problem is overcome by considering adjacent buses tied by a bus tie-breaker as being tied by a very low impedance "line" (e.g., $j0.00001$ per unit impedance).
2. Transformer-magnetizing branch is occasionally required in EHV studies at off-peak hours when the magnetizing current may be a considerable fraction of the total transformer current. If the transformer is located at bus i, inclusion of the magnetizing reactance as a lumped ground tie at bus i is usually adequate.
3. Models for tap-changing transformers and phase-shifters should also include tap and phase-shift limits. These limits may not be symmetrical about the nominal setting. If a tap-changing under load (TCUL) voltage regulating the transformer or phase-shifter hits a limit, that limit should be retained regardless of the results of calculating $\Delta\phi$ and Δt.
4. Transmission line itself is usually modeled as a lumped series impedance and two lumped-shunt capacitive susceptances. The latter occur on each line terminal and represent line-charging capacitance. If the

total line-charging susceptance is jB, a tie of $jB/2$ siemens per unit occurs at each line terminal. The value of B depends on the line configuration and in the case of cables is usually obtained from a computer program.

5. For very long transmission circuits, occasionally the long-line model is used. At 60 Hz, this factor is significant only for the longest lines (e.g., 625 km, which is approximately 1/8 the wavelength). In some, higher frequency signals are involved, and the corresponding shorter 1/8 wavelength point is approximately $37{,}500/f$ km (where f is in hertz). The long-line equations for voltage $V(x)$ and current $I(x)$ at a point x measured from bus 1 are

$$V(x) = V_1 \cosh \gamma x - I_1 Z_c \sinh \gamma x \tag{2.46}$$

$$I(x) = -I_1 \cosh \gamma x + \frac{V_1}{Z_c} \sinh \gamma x, \tag{2.47}$$

where
subscript 1 denotes bus 1
γ and Z_c are the propagation constant and characteristic impedance of the line, respectively

The sign convention in Equations 2.46 and 2.47 is such that $+I_1$ flows in the direction of $+x$. Let bus 2 be the other line terminal located at $x = l$, and let the voltage at bus 2 be V_2 and the current be $+I_2$,

$$\begin{aligned} V_2 &= V_1 \cosh \gamma l - I_1 Z_c \sinh \gamma \ell \\ I_2 &= -I_1 \cosh \gamma l - \frac{V_1}{Z_c} \sinh \gamma \ell. \end{aligned} \tag{2.48}$$

Solving for the currents in terms of voltages,

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{Z_c \tanh \gamma l} & \dfrac{-1}{Z_c \sinh \gamma l} \\ \dfrac{-1}{Z_c \sinh \gamma l} & \dfrac{1}{Z_c \tanh \gamma l} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{2.49}$$

is obtained. Equation 2.48 involves a bus admittance matrix, which can be realized by the two-port *pi* section shown in Figure 2.14. The bus 1 to ground tie is the row sum 1, the bus 2 ground tie is identical, and the off-diagonal entry in Equation 2.48 is the negative of the admittance between buses 1 and 2.

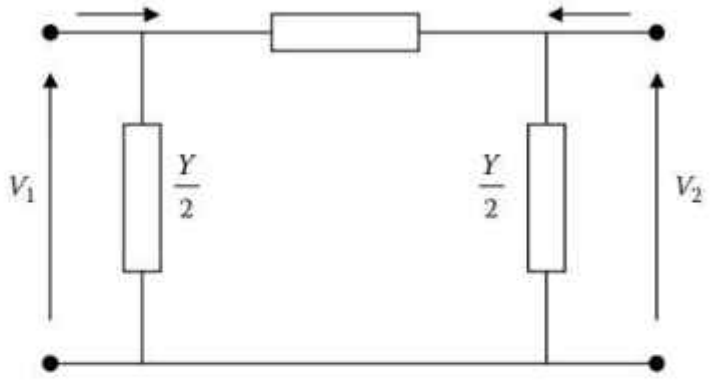


FIGURE 2.14
Simplified network model for an n-generator power system.

2.10 Y-Bus Incorporating the Transformer Effect

The following types of transforms are discussed:

1. Fixed tap transformer
2. Tap-changing under load (TCUL) transformer
3. Phase-shifting transformer

2.10.1 Fixed Tap-Setting Transformer

A transformer with a fixed-tap setting is represented by an impedance or admittance Y_{pq} in series with an ideal autotransformer. For such a transformer at bus p in the line $L_{p\text{-}q}$, we have from Figure 2.12 the transformer ratio:

$$\frac{V_p}{V_t} = \frac{i_{tq}}{I_p} = a. \tag{2.50}$$

Hence

$$I_p = \frac{i_{tq}}{a} = (V_t - V_q)\frac{y_{pq}}{a}. \tag{2.51}$$

But from Equation 2.49 we have

$$V_t = \frac{V_p}{a}. \tag{2.52}$$

Substituting Equation 2.52 into Equation 2.51, we have

$$I_p = (V_p - aV_q)\frac{y_{pq}}{a^2} \tag{2.53}$$

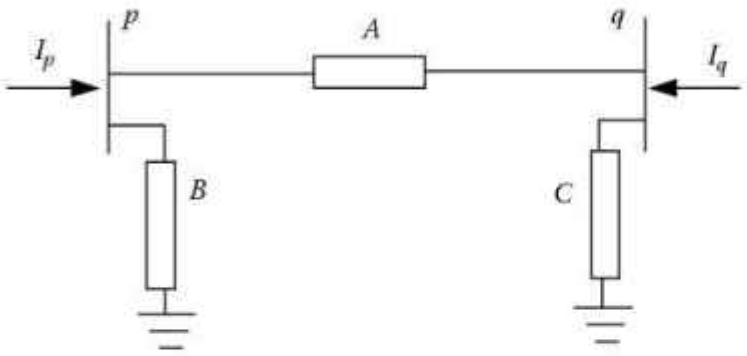


FIGURE 2.15
Equivalent π-circuit of fixed tap transformer.

and also

$$\begin{aligned} I_q &= (V_q - V_t)y_{pq} \\ &= (aV_q - V_p)\frac{y_{pq}}{a}. \end{aligned} \tag{2.54}$$

Such a transformer at bus p in the line p–q can be represented by the equivalent π-circuit shown in Figure 2.15.

The corresponding currents of the equivalent circuit (Figure 2.15) are

$$I_p = (V_p - V_q)A + V_pB \tag{2.55}$$

$$I_q = (V_q - V_p)A + V_qC. \tag{2.56}$$

Now equating the terminal currents I_p and I_q of Figure 2.15, and letting $V_p = 0$, $V_q = 1$, we have from Equations 2.53 and 2.55,

$$I_p = -\frac{y_{pq}}{a} = -A;$$

that is

$$A = \frac{y_{pq}}{a}. \tag{2.57}$$

Similarly, we get

$$I_q = y_{pq} = A + C.$$

Hence

$$\begin{aligned} C &= y_{pq} - A = y_{pq} - \frac{y_{pq}}{a} \\ &= \left(1 - \frac{1}{a}\right)y_{pq}. \end{aligned} \tag{2.58}$$

Similarly for B, we equate terminal current I_p for Equations 2.53 and 2.55. Thus, we get

$$(V_p - aV_q)\frac{Y_{pq}}{a^2} = (V_p - V_q)A + V_pB. \tag{2.59}$$

Substituting for $A = Y_{pq}/a$, and letting $V_p = 1$, $V_q = 0$, we get

$$\begin{aligned} B &= \frac{Y_{pq}}{a^2} - \frac{y_{pq}}{a} = \left[\frac{1}{a^2} - \frac{1}{a}\right] y_{pq} \\ &= \frac{1}{a}\left[\frac{1}{a} - 1\right] y_{pq}. \end{aligned} \tag{2.60}$$

With the substitutions of A, B, and C from Equations 2.57, 2.58, and 2.60, Figure 2.15 takes the form of Figure 2.16.

Thus to represent this transformer in the elements of a Y_{BUS} matrix:

$$\begin{aligned} Y_{pp} &= y_{p1} + \cdots + \frac{y_{pq}}{a} + \cdots + y_{pn} + \frac{1}{a}\left[\frac{1}{a} - 1\right] y_{pq} \\ &= y_{p1} + \cdots + \frac{y_{pq}}{a^2} + \cdots + y_{pn}; \end{aligned} \tag{2.61}$$

that is, this element is changed. Moreover,

$$Y_{pq} = Y_{qp} = -\frac{y_{pq}}{a} \tag{2.62}$$

and also,

$$\begin{aligned} Y_{qp} &= y_{q1} + \cdots + \frac{y_{pq}}{a} + \cdots + y_{qn}, \ \cdots \left[1 - \frac{1}{a}\right] y_{pq} \\ &= y_{q1} + \cdots + y_{qp} + \cdots + y_{qn}; \end{aligned} \tag{2.63}$$

that is, there is no change in this element.

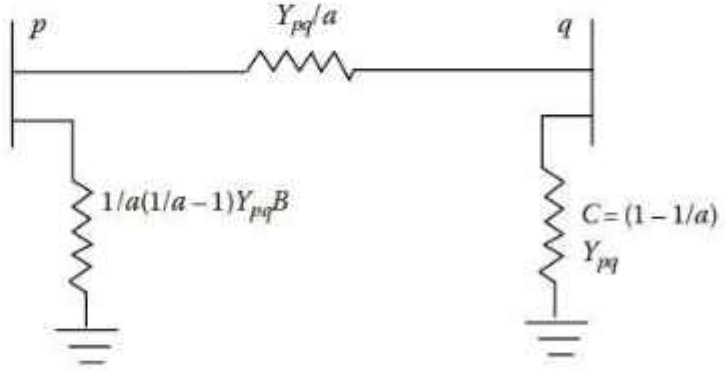


FIGURE 2.16
Equivalent π-circuit of the transformer.

2.10.2 TCUL Transformer

In this case, the taps of the transformer are changed to maintain voltage magnitude within a specified tolerance. Normally, taps are changed once in two iterations and the corresponding values of Y_{pp}, Y_{pq}, and Y_{qq} are calculated such that for any bus p, we have

$$\left|V_p^k - V_p^{\text{scheduled}}\right| < \rho. \tag{2.64}$$

However, to avoid excessive calculations, the series impedance of the equivalent π-circuit is set equal to the series impedance of the transformer and the shunt parameters B and C are changed to reflect the changes in taps.

$$(V_p - V_q)A + V_pB = (V_p - aV_q)\frac{y_{pq}}{a^2} \tag{2.65}$$

with $A = y_{pq}$,

$$(V_p - V_q)y_{pq} + V_pB = (V_p - aV_q)\frac{y_{pq}}{a^2};$$

that is

$$B = \left(\frac{1}{a} - 1\right)\left[\left(\frac{1}{a} + 1\right) - \frac{V_q}{V_p}\right]y_{pq}. \tag{2.66}$$

Similarly taking Equations 2.53 and 2.55 for I_q and substituting $A = y_{pq}$, we have

$$[V_q - V_p]y_{pq} + V_qC = (aV_q - V_p)\frac{y_{pq}}{a}. \tag{2.67}$$

From Equation 2.67, we obtain

$$C = \left(1 - \frac{1}{a}\right)y_{pq}\frac{V_p}{V_q}. \tag{2.68}$$

Subsequently, only the elements Y_{pp} and Y_{qq} are calculated with the change in tappings.

2.10.3 Phase-Shifting Transformer

Tap-changing is a means of voltage control and accompanying reactive-power control. Transformers may also be used to control phase angle and, therefore, active-power flow. Such special transformers are termed phase-shifting transformers or simply phase-shifters. Beyond the limits of rating,

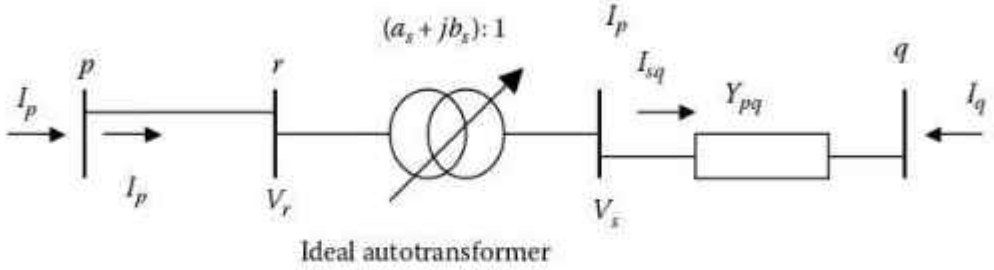


FIGURE 2.17
Phase-shifting transformer model.

tap-setting range and phase-shift range, there are economic considerations. Phase-shifters are costly and there are numerous economically more advantageous alternative methods of active-power control. The principal use of phase-shifters is at major intertie buses where the control of active-power exchange is especially important.

A phase-shifter is a three-phase device, which inserts a voltage V_{psr} into each line. If V_{ser} is out of phase with the line-to-neutral voltage of the source V_{an}, the phase-shifter output voltage V'_{an} will be controllable. Thus the phase of V'_{an} depends on V_{psr}. The magnitude of V'_{an} also varies with V_{psr}. It is possible to construct a controller that will cause only the phase of V'_{an} to vary while keeping the magnitude unchanged. The details of the dependence of $|V'_{an}|$ with the amount of phase shift depends on the phase-shifter design.

A phase-shifting transformer is represented by an ideal transformer with a complex turn ratio $(a + jb)$ in series with impedance or admittance y_{pq} as shown in Figure 2.17. We have $V_p = V_r$ and

$$\frac{V_p}{V_s} = \frac{V_r}{V_s} = a_s + jb_s. \tag{2.69}$$

Since the power loss in the ideal autotransformer is negligible, we have from Figure 2.17

$$V_p^* i_{pr} = V_s^* i_{sq};$$

that is

$$i_{sq}/i_{pr} = V_p^*/V_s^* = a_s - jb_s. \tag{2.70}$$

Now

$$i_{sq} = (V_s - V_q) y_{pq};$$

then

$$i_{pr} = \frac{i_{sq}}{a_s - jb_s} = [(V_s - V_q)] y_{pq}/(a_s - jb_s). \tag{2.71}$$

But since from Equation 3.44,

$$V_s = \frac{V_p}{a_s + jb_s}, \tag{2.72}$$

by substituting Equation 2.73 in Equation 2.72 we obtain

$$i_{pr} = \left[\left(V_p - (a_s + jb_s)V_q\right)\right]\left(y_{pq}/a_s^2 + b_s^2\right). \tag{2.73}$$

Similarly

$$\begin{aligned} i_{qs} &= (V_q - V_s)y_{pq} \\ &= \left[(a_s + jb_s)V_q - V_p\right]\frac{y_{pq}}{a_s + jb_s}. \end{aligned} \tag{2.74}$$

Now we calculate the elements of the Y_{BUS} matrix by conducting a short-circuit test. The diagonal element Y_{pp} is found by connecting a unit voltage source at the pth bus (i.e., $V_p = 1$ pu) and short circuiting the remaining buses (i.e., substituting the voltages at other buses as zero); then we get

$$\begin{aligned} I_p = Y_{pp} &= I_{p1} + I_{p2} + \cdots + i_{pr} \\ &= [V_p - V_1]y_{p1} + [V_p - V_{p2}]y_{p2} + \cdots + i_{pr} \\ &= y_{p1} + y_{p2} + \cdots + \left[V_p - (a_s + jb_s)V_q\right]y_{pq}/a_s^2 + b_s^2. \end{aligned} \tag{2.75}$$

As $V_p = 1$ pu and $V_i = 0$ for $i = 1, \ldots, n$, $i \neq p$, we get

$$I_p = Y_{pp} = y_{p1} + y_{p2} + \cdots + \left(y_{pq}/a_s^2 + b_s^2\right) + \cdots; \tag{2.76}$$

that is, this element is changed.

Similarly for Y_{qq}, we let $V_q = 1$ pu and the rest of the bus voltages be set to zero; we get

$$\begin{aligned} I_q = Y_{qq} &= I_{q1} + I_{q2} + \cdots + I_{qS} \\ &= [V_q - V_1]y_{q1} + [V_q - V_2]y_{q2} + \left((a_s + jb_s)V_q - V_p\right)y_{qq}/a_s + jb_s. \end{aligned} \tag{2.77}$$

Since $V_q = 1$ pu and the rest of the bus voltages viz. $V_1 = V_2 \cdots = 0$, we get

$$I_q = Y_{qq} = y_{qi} + y_{q2} + \cdots + y_{pq}, \tag{2.78}$$

that is no change.

For mutual admittances, say Y_{qp}, we apply $V_p = 1$ pu and measure I_q keeping other bus voltages equal to zero (i.e., $I_q = Y_{qp}V_p$) but since $V_p = 1$ pu we get from Equation 2.75,

$$I_q = Y_{qp} = -\frac{y_{pq}}{a_s + jb_s}. \tag{2.79}$$

For Y_{pq} we keep $V_q = 1$ and the rest of the buses are short circuited; then from Equation 2.75 we get

$$I_p = Y_{pq} = -\frac{y_{pq}}{a_s - jb_s}. \tag{2.80}$$

Moreover, $Y_{pq} \neq Y_{qp}$. Now the complex turn ratio $a_s + jb_s = a[\cos\theta + j\sin\theta]$ where

$$|V_p| = a|V_s|. \tag{2.81}$$

However, if θ is positive, the phase of $|V_p|$ is advanced, that is, leading with respect to that of $|V_s|$ or V_q.

Figure 2.18 shows a generalized model of a phase-shifter in which both voltage magnitude and phase angle vary. The equivalent tap position is $t(\phi)$ which is only a function of the phase setting ϕ. If no voltage variation occurs, $t = 1$. The phase-shifter admittance is Y_{ij}. Figure 2.19 shows a further equivalent. Note that in Figure 2.19

$$|V_i| = |V_i'|$$
$$\delta_{i'} = \delta_i + \phi$$

and the complex volt–amperes entering the phase-shifter at bus i are delivered to bus i'. Let bus i' be introduced into the $[\delta, V]^T$ vector, but only in the δ

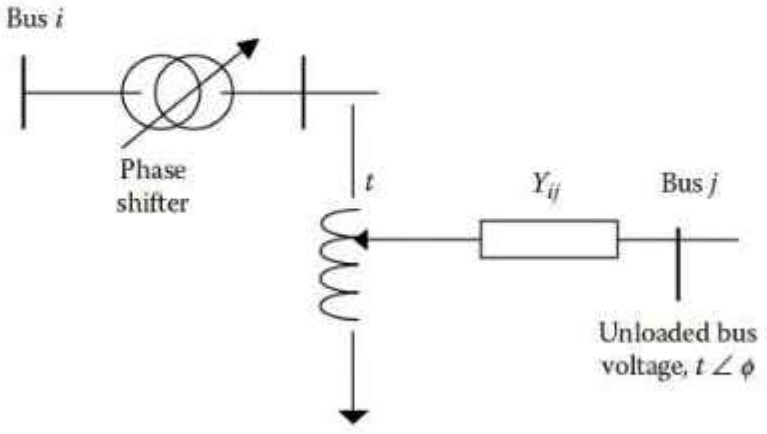


FIGURE 2.18
Generalized model of a phase-shifter.

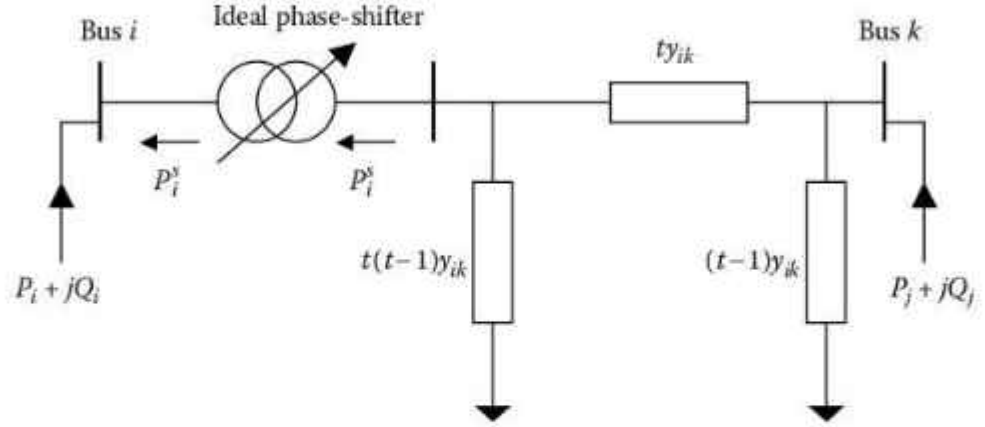


FIGURE 2.19
Equivalent circuit of a phase-shifter.

subvector. Also, rather than writing $\delta_{i'}$ in this subvector, we introduce the unknown phase shift ϕ:

$$[\delta_2, \ldots, \delta_N, \phi_1, \ldots, \phi_N, V_2, \ldots, V_i, V_{i+1}, \ldots, V_N]^T.$$

Note that V_i' is missing from the $|V_{\text{bus}}|$ subvector. Also note that the partial derivatives $\partial P/\partial \delta_{i'}$ and $\partial Q/\partial \delta_{i'}$ are equal to $\partial P/\partial \phi$ and $\partial Q/\partial \phi$, respectively, since $\delta_i' = \delta_i + \phi$. The missing voltage magnitude element is counterbalanced by a missing $\Delta Q_i'$ expression. Thus the size of J is $2(N-1) - n_\phi$ by $2(N-1) - n_\phi$, where n_ϕ is the number of phase-shifters in the network.

The procedure for handling a phase-shifter bus where ϕ adjusts itself to give P_i^ϕ (see Figure 2.18) injected into bus i is as follows:

1. Construct the Y_{bus} using the conventional algorithm without connecting buses i' and i, or buses i' and k.
2. Use the equivalent circuit in Figure 2.19 to connect buses i' and k. Use the initialized value of ϕ to find t. The equivalent tap-setting is a function of ϕ in general (this depends on the phase-shifter design).
3. Construct the Jacobian matrix. Eliminate the ΔQ expression at bus i and eliminate $|V|$ at bus i.
4. Consider P_i^ϕ the specified power through the phase-shifter, as injected into i and as the only load at i'.
5. Perform one iteration in the power flow study and repeat as required.

To determine the effect of changes in real and reactive generation sources on the overall power flow, the simplest approach is by trial and error. The generators, power, and voltage levels are maintained to match the desired load and voltage profile. A power-flow analysis is made and the corresponding flow picture is examined. If equipment loadings are outside their desired

limits, adjustments are made to the set points and a new power-flow analysis is performed. It is hoped that the procedure would converge upon a suitable power-flow picture iteratively.

2.11 Load Models [3]

The prespecified S load (i.e., independent of bus voltage) is a reasonable representation for buses with a large percentage of rotating machines, loads, and voltage-regulated loads. Other alternative load models include the following:

1. Loads whose $|S|$ demand is proportional to $|V|^2$. This type of load is a prespecified impedance load and is common when the load contains significant incandescent lighting and resistive heating.
2. Loads whose $|S|$ demand is proportional to $|V|^1$. This type of load is a prespecified current load and is common when the load contains significant rectifier loads and certain types of synchronous machine loads.

It has been suggested that the actual load be modeled as a linear combination of each of the load types mentioned (constant S load in which $|S| \alpha |V|^0$, constant current load in which $|S| \alpha |V|^1$, and constant impedance load in which $|S| \alpha |V|^2$). Such an approach would require considerable knowledge of the load composition or a knowledge of the active and reactive-power variation with $|V|$ (from a historical perspective). Load models are traditionally classified into two broad categories: static models and dynamic models.

2.11.1 Static Load Models

A static load model expresses the characteristic of the load at any instant of time as algebraic functions of the bus voltage magnitude and frequency at that instant. The active power component P and the reactive-power component Q are considered separately.

The load characteristics in terms of voltage are represented by the exponential model:

$$\begin{aligned} P &= P_0\left(\frac{V}{V_0}\right)^a \\ Q &= Q_0\left(\frac{V}{V_0}\right)^a, \end{aligned} \tag{2.82}$$

where

P and Q are active and reactive components of the load when the bus voltage magnitude is V

Subscript "0" identifies the values of the respective variables at the initial operating condition

The parameters of this model are the exponents a and b. With these exponents equal to 0, 1, or 2, the model represents constant power, constant current, or constant impedance characteristics, respectively. For composite loads, their values depend on the aggregate characteristics of load components.

An alternative model that has been widely used to represent the voltage dependence of loads is the polynomial model:

$$P = P_0\left[p_1\left(\frac{V}{V_0}\right)^a + p_2\left(\frac{V}{V_0}\right) + p_3\right]$$
$$Q = Q_0\left[q_1\left(\frac{V}{V_0}\right)^a + q_2\left(\frac{V}{V_0}\right) + q_3\right]. \tag{2.83}$$

This model is commonly referred to as the ZIP model, since it is composed of constant impedance (Z), constant current (I), and constant power (P) components. The parameters of the model are the coefficients P_1–P_3 and Q_1–Q_3 which define the proportion of each component.

Multiplying the exponential model or the polynomial model as follows represents the frequency dependence of load characteristics by a factor.

$$P = P_0\left(\frac{V}{V_0}\right)^a(1 + K_{pf}\Delta f)$$
$$Q = Q_0\left(\frac{V}{V_0}\right)^a(1 + K_{qf}\Delta f) \tag{2.84}$$

or

$$P = P_0\left[p_1\left(\frac{V}{V_0}\right)^a + p_2\left(\frac{V}{V_0}\right) + p_3\right](1 + K_{pf}\Delta f)$$
$$Q = Q_0\left[q_1\left(\frac{V}{V_0}\right)^a + q_2\left(\frac{V}{V_0}\right) + q_3\right](1 + K_{qf}\Delta f), \tag{2.85}$$

where Δf is the frequency deviation, given as $(f - f_0)$.

2.12 Available Transfer Capability

The introduction of competition in the power industry will increase the search for better utilization of transmission facilities. A necessary condition to support competition is open access by market participants to the transmission network. Several open access schemes are currently being discussed. Transmission companies are facing several challenges in the new economic environment including the assessment of adequate transmission charges and the choice of the best transmission investment options. These aspects have motivated the development of methodologies to evaluate existing power transfer capabilities and transmission margins. Under FERC Orders 888 and 889, all transmission providers must determine and offer for sale ATC taking into account existing obligations and allowing appropriate margins to maintain reliability. There is a growing body of work dealing with determination of ATC. The main intention is to evaluate the ability of a network to allow for the reliable movement of electric power from areas of supply to areas of demand. The framework of analysis may or may not take into account the stochastic nature of loads and equipment availability. Also, either DC or AC models can assess the performance of the electric network.

2.12.1 ATC Definition and Formulation

Total transfer capability (TTC) determines the amount of electric power that can be transferred over the interconnected transmission network in a reliable manner based on all the following conditions.

For the existing or planned system configuration, and with normal (precontingency) operating procedures in effect, all facility loadings must be within normal ratings and all voltages must be within normal limits. The electric systems must be capable of absorbing the dynamic over swings, and remain stable, following a disturbance that results in the loss of any single electric system element, such as a transmission line, transformer, or generating unit. Dynamic power savings subside following a disturbance that results in the loss of any single electric system element as described and after operation of any automatic operating systems. But before any postcontingency operator-initiated system adjustments are implemented, all transmission facility loadings should be within emergency ratings and all voltages within emergency limits. When precontingency facility loadings reach normal thermal ratings at a transfer level below that at which any first contingency transfer limits are reached, the transfer capability is defined as the transfer level at which such normal ratings are reached. In some cases depending on the geographical area, specified multiple contingencies may be required to determine the transfer capability limits. If these limits are more restrictive than the single contingency limits, they should be used. With this

very general definition, we note that the TTC is a function of system thermal, voltage, and transient stability limits and is given by

$$\text{TTC} = \min[\text{thermal limits, voltage limits, transient stability limits}].$$

Transmission reliability margin (TRM) is the amount of transfer capability necessary to ensure that the interconnected transmission network is secure under a reasonable range of system conditions. This measure reflects the effect of various sources of uncertainty in the system and in system operating conditions. It also captures the need for operational flexibility in ensuring reliable and secure system operations.

Capacity benefit margin (CBM) is the amount of transfer capability reserved by load-serving entities to ensure access to generation from interconnected neighboring systems to meet generation reliability requirements.

ATC then is defined to be a measure of the transfer capability remaining in the physical transmission network for further commercial activity over and above already committed uses. It is given by the relationship

$$\text{ATC} = \text{TTC} - \text{TRM} - \text{CBM} - \text{existing transmission commitments}.$$

FERC requirements call for the calculation and posting (on the OASIS) of continuous ATC information for the next hour, month, and for the following 12 months.

Several regions have already started calculating and posting ATCs. However, there are certain inconsistencies in their calculation that need to be addressed:

- Use of real-time conditions for ATC calculation.
- Use of accurate power flow methods to calculate ATC rather than DC power flows or linear interpolation techniques.
- Power system flows are not independent.
- Individual users do not coordinate their transmission system use.
- Limiting effects on a transmission system are nonlinear functions of the power flow.

2.12.2 ATC Calculation

The ATC problem is the determination of the largest additional amount of power above some base case value that can be transferred in a prescribed manner between two sets of buses: the source, in which power injections are decreased by an offsetting amount. Increasing the transfer power increases the loading in the network, and at some point causes an operational or physical limit to be reached that prevents further increase. The effects of contingencies are taken into account in the determination. The largest value

of transfer power that causes no limit violations, with or without a contingency, is used to compute the TTC and ATC.

Limits checked for the base case are normal branch flow limits, normal corridor transfer limits, bus voltage limits, and voltage collapse condition. For contingency cases, the limitations checked are the emergency branch flow limits, emergency corridor transfer limits, bus voltage limits, bus change limits, and voltage collapse condition.

The following is a summary of the steps for determining the ATC for a list of source/sink transfer cases for one time period, usually 1 h.

1. Establish and solve the base case power flow for the time period.
2. Select a transfer case.
3. Use continuation power flow (CPF) to make a step increase in transfer power.
4. Establish a power flow problem consisting of the base case modified by the cumulative increases in transfer power from step 3.
5. Solve the power flow problem of step 4.
6. Check the solution of step 5 for violations of operational physical limits.
7. If there are violations, decrease the transfer power to the minimum amount necessary to eliminate them.
8. Solve the power flow problem of steps 4–6 for each listed contingency.
9. If there are violations, decrease the transfer power by the minimum amount necessary to eliminate them. This is the maximum transfer power for the case.
10. Compute the ATC from the interface flows in the adjusted solution of step 7.
11. Return to step 2 for the next transfer case.

2.13 Illustrative Examples

Example 2.13.1

The system in Figure 2.20 is known to be balanced and is a negative sequence network. Assuming that

$$Z_{\text{line}} = 10.0\angle -15^\circ\ (\Omega) \quad \text{and} \quad V_{\text{ac}} = 208\angle -120^\circ\ \text{V}$$

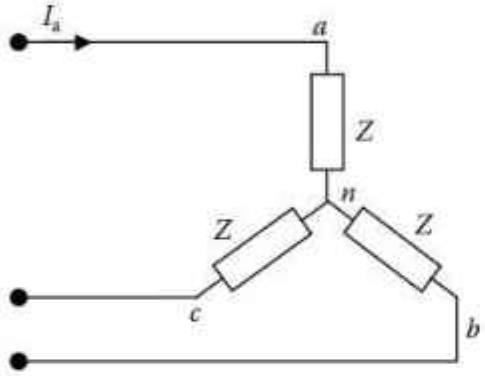


FIGURE 2.20
Y-connected network for Example 2.13.1.

find

1. Voltages V_{ac}, V_{bc}, V_{bn}, V_{cn}
2. Currents I_a, I_b, and I_c
3. Total three-phase power $S_{3\phi}$

SOLUTION

1. Line voltages are

$$\overline{V}_{ca} = -\overline{V}_{ac} = 208\angle -120^\circ + 180 = 208 = \angle 60^\circ \text{ V}$$
$$\overline{V}_{ab} = \overline{V}_{ac}\angle 120^\circ = 208\angle 60^\circ + 120^\circ = 208\angle 180^\circ \text{ V}$$
$$\overline{V}_{bc} = \overline{V}_{ab}\angle -240^\circ = 208\angle 180^\circ - 240^\circ = 208\angle -60^\circ \text{ V}.$$

Phase voltages are

$$\overline{V}_{an} = \frac{\overline{V}_{ab}}{\sqrt{3}}\angle 30^\circ = \frac{208}{\sqrt{3}}\angle 180^\circ + 30^\circ = 120\angle 210^\circ \text{ V}$$
$$\overline{V}_{bn} = \overline{V}_{an}\angle -240^\circ = 120\angle 210^\circ - 240^\circ = 120\angle -30^\circ \text{ V}$$
$$\overline{V}_{cn} = \overline{V}_{bn}\angle 120^\circ = 120\angle -30^\circ + 120^\circ = 120\angle 90^\circ \text{ V}.$$

2. Line current

For the Y-connected network, the line current = phase current,

$$\overline{I}_a = \frac{\overline{V}_{an}}{Z_{line}} = \frac{120\angle 210^\circ}{10\angle -15^\circ} = 12\angle 195^\circ \text{ V}$$
$$\overline{I}_b = \overline{I}_a\angle -240^\circ = 12\angle 195^\circ - 240^\circ = 12\angle -45^\circ \text{ V}$$
$$\overline{I}_c = \overline{I}_b\angle 120^\circ = 12\angle -45^\circ + 120^\circ = 12\angle 75^\circ \text{ V}.$$

3. Three-phase power

$$S_{3-\phi} = \sqrt{3}*208*12*\cos(15^\circ) = 4175.9 \text{ W}$$

Example 2.13.2

In Figure 2.21, find the total complex power delivered to the load. Assume that

$$Z_c = -j0.2\ \Omega$$
$$Z_L = +j0.1\ \Omega$$
$$R = 10\ \Omega.$$

Solution

$$Y_{\text{equivalent}} = \frac{1}{R} + \frac{1}{Z_c} + \frac{1}{Z_L} = \frac{1}{10} + \frac{1}{-j0.2} + \frac{1}{j0.1}$$
$$= 0.1 + j5 - j10 = (0.1 - j5)\ \Omega.$$

The total complex power is defined as $S = VI^* = V(VY)^*$. Or we can say $S^* = V^*I = V^*(VY) = V^2Y = (1.0)^2(0.1 - j5.0)$. The total complex power is $S = 0.1 + j5$ VA.

Example 2.13.3

You are given two system areas connected by a tie line with the following characteristics.

Area 1	Area 2
$R = 0.01$ pu	$R = 0.02$ pu
$D = 0.8$ pu	$D = 1.0$ pu
Base MVA = 2000	Base MVA = 1000

A load change of 100 MW occurs in area 2. Assume both areas were at nominal frequency (60 Hz) to begin

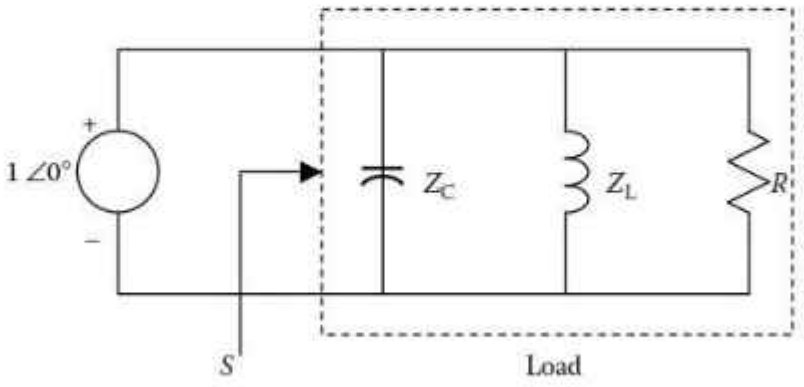


FIGURE 2.21
Composite R–L–C Load for Example 2.13.1

1. What is the new steady-state frequency in Hertz?
2. What is the change in tie-line flow in megawatt?

SOLUTION

Based on the information in Section 2.8, if several generators (each having its own governor and prime mover) were connected to the system, the frequency droop characteristics β_1 and β_2 are given by

$$\beta_1 = \left(\frac{1}{R_1} + D_1\right); \quad \beta_2 = \left(\frac{1}{R_2} + D_2\right)$$

$$\Rightarrow \beta_1 = \frac{1}{0.01} + 0.8 = 100.8 \text{ pu} \quad \text{and} \quad \beta_2 = \frac{1}{0.02} + 1.0 = 51.0 \text{ pu}.$$

Using a common base of 1000 MVA, the frequency droop characteristics becomes

$$\beta_1 = \frac{100.8 \times 2000}{1000} = 201.6 \text{ pu} \quad \text{and} \quad \beta_2 = \frac{51.0 \times 1000}{1000} = 51.0 \text{ pu}.$$

Calculating the change in frequency based on the load change in area 2

$$\Delta\omega = \frac{-\Delta P_{L_2}}{\beta_1 + \beta_2} = \frac{-\Delta P_{L_2}}{(1/R_1) + (1/R_2) + D_1 + D_2}.$$

The change in load is given as $\Delta P_{L_2} = 100 \text{ MW} = 100/1000 = 0.1$ pu.

Hence, $\Delta\omega = \dfrac{-0.1}{201.6 + 51.0} = -0.00039588$ pu.

(a) We have $\dfrac{\Delta\omega}{\omega_0} = \dfrac{\Delta f}{f_0} = -0.00039588$ pu

$$\Rightarrow \Delta f = 60(-0.00039588) = -0.023753 \text{ Hz}$$

and the new frequency becomes

$$f = 60 - 0.023753 = 59.976 \text{ Hz}.$$

(b) Change in tie flow is given by

$$\Delta P_{\text{tie}} = -\beta_1 \Delta\omega = \frac{-\Delta P_{L_2}((1/R_1) + D_1)}{(1/R_1) + (1/R_2) + D_1 + D_2} = -(201.6)(-0.00039588)$$

$$= 0.07981 \text{ pu}$$

$$\Rightarrow \Delta P_{\text{tie}} = 0.07981 \text{ pu} \times 1000 \text{ MW} = 79.81 \text{ MW}.$$

This analysis shows that 79.81 MW of power is being supplied from area 1 to area 2 to meet the 100 MW load increase in area 2.

2.14 Conclusions

The chapter explained major basics for modeling different components in typical electric power systems. First, the concept of electric power was introduced expressed in terms of active, reactive, and apparent power. Second, a review of three-phase systems was introduced. Finally, the synchronous machine model from an electric network standpoint was presented; the reactive capability curves are also examined. Furthermore, the model was extended to handle the excitation system governor models. Static and dynamic load models were also discussed.

2.15 Problem Set

PROBLEM 2.15.1

In Figure 2.22, assume that

$$|V_1| = |V_2| = 1.00 \text{ pu}$$
$$Z_{\text{line}} = 0.1\angle 85^\circ.$$

1. For what nonzero θ_{12} is S_{12} purely real?
2. What is the maximum power, $-\mathbf{P}_{12}$, that can be received by V_2, and at what θ_{12} does this occur?
3. When $\theta_{12} = 85^\circ$, what is the active power loss in the line?
4. For what value θ_{12} is $-\mathbf{P}_{12} = 1.00$ pu?

PROBLEM 2.15.2

Draw the power circle diagram in the case $|V_1| = 1.05$, $|V_2| = 0.95$, $Z_{\text{line}} = 0.1$ pu at 85°. Find

1. $\mathbf{P}_{12(\max)}$
2. θ_{12} at which we get $\mathbf{P}_{12(\max)}$
3. $-\mathbf{P}_{12(\max)}$
4. θ_{12} at which we get $-\mathbf{P}_{12(\max)}$
5. Active power loss in the line when $\theta_{12} = 10^\circ$



FIGURE 2.22
Simple two-bus power system network for Problem 2.15.1.

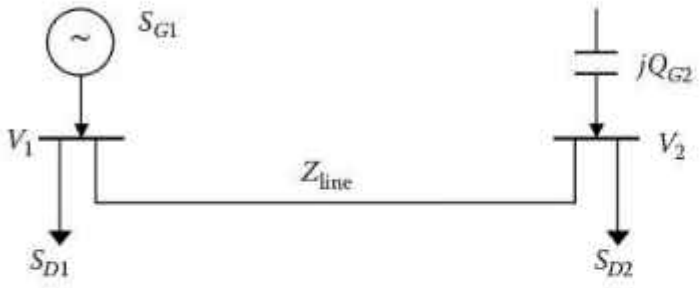


FIGURE 2.23
Power system model for Problem 2.15.3.

PROBLEM 2.15.3

In Figure 2.23, assume that

$$V_1 = 1.00\angle 0^\circ \text{ pu}$$

Given that

$$Z_{\text{line}} = 0.10 + j0.10$$
$$S_{D1} = 0.50 + j0.50$$
$$S_{D2} = 0.50 + j0.50,$$

select Q_{G2} such that $|V_1| = 1.00$ pu. In this case what are Q_{G2}, S_{G1}, and arg V_2?

PROBLEM 2.15.4

A three-phase, Y-connected synchronous generator is rated for a 600 kVA and 6.9 kV. The reactance of the stator winding is $X_s = j3.97\ \Omega$/phase. Express X_s in per units on the machine ratings as its bases.

PROBLEM 2.15.5

A three-phase, 500 kVA, 6.9 Y/1.2 Y–kV transformer has series impedance referred to the low-voltage side of $Z_{T_2} = 0.72\angle 80^\circ\ \Omega$/phase. Express Z_T in per unit.

PROBLEM 2.15.6

A three-phase synchronous generator having $R_a = 0$ and $X_S = 1.0$ pu is operating as a generator with $V_t = 1.00$ pu, pf = 1.0, and power delivered = 1.00 pu.

1. Determine E_{F1} and δ_1, for this operation.
2. Obtain I_a, pf_2, and δ_2 for $E_{F2} = 1.6$ pu with P = constant.



FIGURE 2.24
A two-area power system interface for Problem 2.15.8.

PROBLEM 2.15.7

Consider a load connected across an AC voltage source where the load impedance is given by $R_L + jX_L$ and the series source impedance is given by $R_S + jX_S$. Determine the optimum values for R_L and X_L for maximum power transfer to the load.

PROBLEM 2.15.8

Consider the two area systems shown in Figure 2.24. The system characteristics are given in the figure.

$$P_{G_0}^{A} = P_L^{A} = 1500 \text{ MW}$$
$$R_A = 0.015$$
$$P_{G_0}^{A} = P_L^{B} = 5000 \text{ MW}$$
$$R_B = 0.008.$$

Assume a load increase of 100 MW occurs in area 1; determine the change in frequency and the ACE in each area.

PROBLEM 2.15.9

Given the block diagram of the two interconnected areas shown in Figure 2.25.

1. Derive the transfer function that relates $\Delta W_1(s)$ and $\Delta W_2(s)$ to load change $\Delta P_L(s)$.
2. For the following data (base power is 1000 MVA)

$$M_1 = 3.8 \text{ pu}, \quad D_1 = 0.9 \text{ pu}, \quad f_1^{o} = 60 \text{ Hz}$$
$$M_2 = 4.0 \text{ pu}, \quad D_2 = 0.8 \text{ pu}, \quad f_2^{o} = 60 \text{ Hz}$$
$$T = 7.6 \text{ pu}, \quad \Delta P_L = 200 \text{ MW}, \quad \text{and} \quad R = 0.05 \text{ pu},$$

calculate the final frequency.

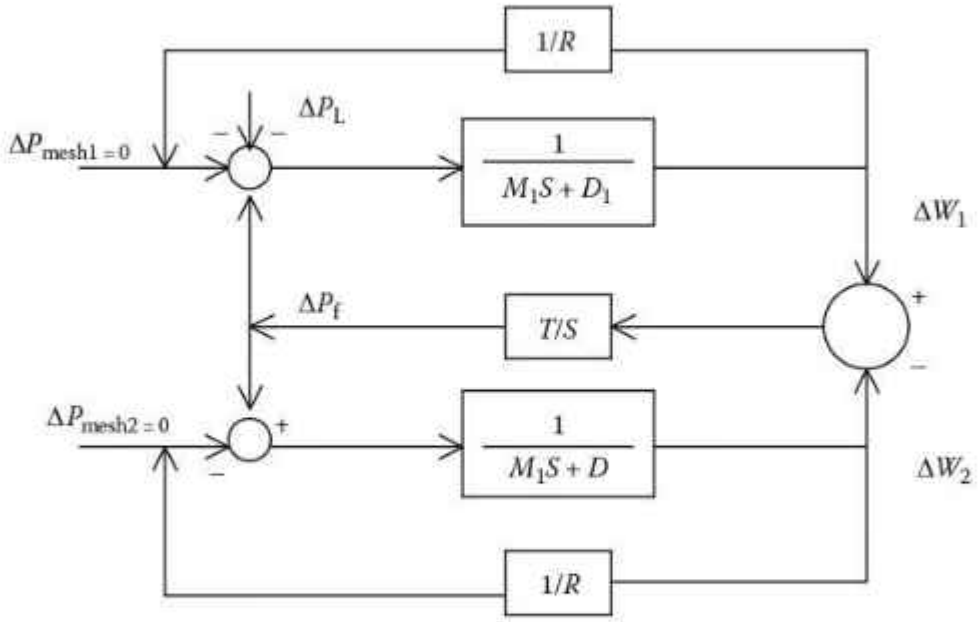


FIGURE 2.25
Block diagram for the control of two interconnected areas for Problem 2.15.9.

References

1. Bergen, A. R. and Vittal, V. *Power Systems Analysis*, 2nd edn., Prentice-Hall, Upper Saddle River, NJ, 1999.
2. Elgerd, O. I. *Electrical Energy Systems Theory—An Introduction*, McGraw-Hill, New York, 1982.
3. El-Hawary, M. E. *Electric Power Systems*, 2nd edn., IEEE Press, New York, 1995.
4. Fizgerald, A., Kingsley, L., and Urans, S. *Electric Machinery*, McGraw-Hill, New York, 1983.
5. Fouad, A. A. and Vittal, V. *Power System Transient Stability; Analysis in Transient Energy Function Method*, Prentice-Hall, Englewood Cliffs, NJ, 1992.
6. Momoh, J. A. and El-Hawary, M. E. *Electrical Systems, Dynamics and Stability with Artificial Intelligence Applications*, Marcel Dekker, New York, 1999.
7. Sauer, P. W. and Pai, M. A. *Power System Dynamics and Stability*, Prentice-Hall, Upper Saddle River, NJ, 1998.

3

Power-Flow Computations

3.1 Introduction

Power flow (PF) is an evaluation tool for operation and planning for determining the steady condition of operation [1,2]. As there is a lot of enhancement of the computation for power system performance, we need to assure the capability of PF to be guaranteed, stressed nature for congestion stresses. The feasibility of PF solution is a necessary condition for pricing theory and stability analysis as well as further robust optimal power flow (OPF) program. Therefore stability for necessity of the performance of PF tools becomes very important.

We must monitor and provide measures to achieve convergence through

- Appropriate selection of initial condition
- Handling of negative impedances
- Location of the swing bus
- Computation of losses
- Managing congestion and power balance under different operating conditions
- Error handling and alert messages
- Final interpretation of results, especially for nonconvergence of the PF methods

It should be noted that PF is a well-seasoned tool used by the industry and researchers. Several packages are available in the market for easy access and study both for industry and education [3].

PF studies are routinely used in planning, control, and operations of existing electric power systems as well as planning for future expansion. Satisfactory operation of power systems depends upon knowing the effects of adding interconnections, connecting new loads, introducing new generating stations, or constructing new transmission lines before they are installed. PF studies also allow us to determine the best size and the most favorable locations for power capacitors both for improving the power factor and also raising the bus voltages of the electrical network. PF studies help us

determine the best location as well as the optimal capacity of proposed generating stations, substations, or new lines.

In power system operations, a PF study is used to evaluate the state of the system under a given operational condition and to allow the operator to detect overloads and evaluate transfer limits. Contingency analysis relies on load-flow studies designed to test for the effects of line or generator outages. The information obtained from the load-flow studies includes the magnitude and phase angle of voltages at each bus and the active and reactive PF in each line.

The extensive calculations required for both PF voltage determination requires the use of specialized computing tools [3–6]. This led to the design of a special purpose analog computer called the AC network analyzer in 1929. An AC network analysis was capable of simulating the operation of the power system under existing conditions as well as proposed future expansion. Programming digital computers for PF studies gained importance, and the first computer-based planning studies were completed in 1956. This change from the network analyzer to the digital computer has resulted in greater flexibility, economy, accuracy, and faster operation.

3.2 Types of Buses for PF Studies

The buses in an electric power system network are generally divided into three categories: generation bus, load bus, and slack bus, and two of the following quantities are specified at each bus.

1. Magnitude of the voltage, $|V|$
2. Phase angle of the voltage, ϕ
3. Active or real power, P
4. Reactive power, Q

The quantities specified at each of the bus types are:

1. Generation bus (or voltage-controlled bus): This is also called the P–V bus, where the voltage magnitude $|V|$ and real power P are specified.
2. Load bus: This is also called the P–Q bus, where the real power P and reactive power Q are specified.
3. Slack or swing bus: This is also known as the reference bus where the voltage magnitude $|V|$ and phase angle ϕ are specified. This bus is selected to provide additional real and reactive power to supply transmission losses since these are unknown until the final solution is obtained. If the slack bus is not specified, then a generation bus (usually with maximum real power P) is taken as the slack bus. There can be more than one slack bus in a given system.

TABLE 3.1

Typical Load-Flow Bus Specifications

	Bus Parameters				
Bus Classification	**Real Power P_i**	**Reactive Power Q_i**	**Voltage Magnitude $\|V\|$**	**Voltage Argument Φ**	**Comments**
Slack or reference bus			√		Must adjust net power to hold voltage constant
Voltage controlled or P–V bus			√		$Q_{min} \leq Q \leq Q_{max}$
Load or P–Q bus	√				Constant power, constant impedance, and constant current

Table 3.1 shows the load-flow bus specifications. The checked quantities are the boundary conditions for the particular bus.

PF analysis is concerned not only with the actual physical mechanism that controls the PF in the network, but also with how to select a best or optimum flow configuration from among the myriad possibilities [1–6]. The following points summarize the more important features incorporated in PF analysis:

1. Transmission links can carry only certain amounts of power; therefore, links must not be operated too close to their stability or thermal limits.
2. It is necessary to keep the voltage levels of certain buses within close tolerances. This can be achieved by proper scheduling of reactive power levels.
3. If the power system is part of a larger pool, it must fulfill certain contractual commitments via its tie-lines to neighboring systems.

The overall PF problem can be divided into the following:

1. Formulation of a suitable mathematical network model: The model must describe adequately the relationships among voltages and powers in the interconnected system.
2. Specification of the power and voltage constraints that must apply to the various buses of the network.
3. Numerical computation of the PF equations (PFEs) subject to specified constraints. These computations provide, with sufficient accuracy, the values of all bus voltages.
4. When all bus voltages have been determined, the actual PFs in all transmission links are computed.

3.3 General Form of the PFEs

To understand PF in an electric power system, a mathematical model from which the flow picture can be predicted must be established. This model is the PFE. In the general case, consider the general bus i shown in Figure 3.1. Its generation and load are assumed to be equal, S_{Gi} and S_{Di}, respectively. The bus power S_i is thus given by

$$S_i = S_{Gi} - S_{Di} = P_{Gi} - P_{Di} + j(Q_{Gi} - Q_{Di}). \tag{3.1}$$

Transmission lines connect bus i to other buses k in the system. At most there can be $n-1$ such outgoing lines, each of which can be represented by a π equivalent, with series and parallel admittances y_{sik} and y_{pik}, respectively. If a line does not exist, its admittances simply are set at zero. The current balanced equation is

$$\begin{aligned} I_i = \frac{S_i^*}{V_i^*} &= V_i \sum_{\substack{k=1 \\ k\neq 1}}^{n} y_{pik} + \sum_{\substack{k=1 \\ k\neq 1}}^{n} y_{sik}(V_i - V_k) \\ &= V_i \sum_{\substack{k=1 \\ k\neq 1}}^{n} y_{sik}(V_i - V_k) + \sum_{\substack{k=1 \\ k\neq 1}}^{n} (-y_{sik})V_k, \quad \text{for } i = 1, 2, 3, \ldots, n. \end{aligned} \tag{3.2}$$

These equations can obviously be written in the form

$$I_i = \frac{S_i^*}{V_i^*} = Y_{i1}V_1 + Y_{12}V_2 + \cdots Y_{ii}V_i + \cdots Y_{in}V_n, \quad \text{for } i = 1, 2, \ldots, n, \tag{3.3}$$

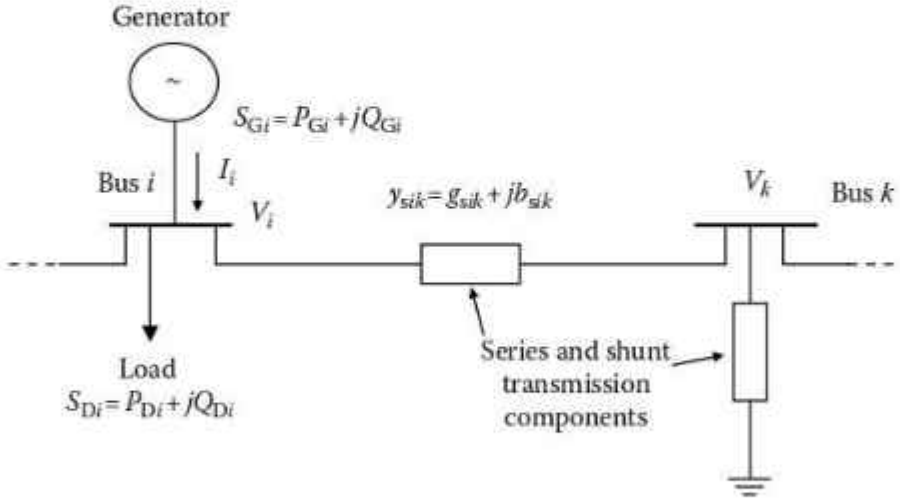


FIGURE 3.1
General bus with generation, load, and outgoing lines.

where

$$Y_{ii} \triangleq \sum_{\substack{k=1 \\ k \neq i}}^{n} (y_{pik} + y_{sik}) \tag{3.4}$$

and

$$Y_{ik} = Y_{ki} \triangleq -\frac{1}{2} y_{sik}. \tag{3.5}$$

The n-dimensional bus current and bus voltage vectors are then introduced:

$$\begin{aligned} \mathbf{I}_{\text{bus}} &= \begin{bmatrix} I_1 \\ \vdots \\ I_n \end{bmatrix} \\ \mathbf{V}_{\text{bus}} &= \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} \end{aligned} \tag{3.6}$$

and the $n \times n$-dimensional bus admittance and bus impedance matrices are

$$\mathbf{Y}_{\text{bus}} = \left[Y_{ij}\right]_{n \times n}; \quad \mathbf{Z}_{\text{bus}} = \mathbf{Y}_{\text{bus}}^{-1} = \left[Z_{ij}\right]_{n \times n}. \tag{3.7}$$

3.3.1 PF Control by Transformer Regulation

To employ transformer ratio or angle setting for PF control we proceed to choose transformer ratios and angular settings that appear to best meet the specified requirements. By analyzing the PF results, one finds a basic difference between this type of control and source control.

Adjusting transformer settings will affect the $\mathbf{Y}_{\text{bus}}$ matrix, while changes in generation level will not change the $\mathbf{Y}_{\text{bus}}$ matrix. Consider a two-bus system, with a regulating transformer (RT) at bus 2 as shown in Figure 3.2a. The RT has a complex transformation ratio

$$a = |a| \angle \alpha. \tag{3.8}$$

RT is represented as an ideal transformer and its effects upon the $\mathbf{Y}_{\text{bus}}$ matrix are calculated. From Figure 3.2b, it can be observed that the voltage and current on the primary side of RT are equal to aV_2 and I_2/a^*, respectively. V_2 and I_2 are the secondary variables. The current balance at the two buses requires that

$$I_1 = \frac{S_1^*}{V_1^*} = V_1 Y_p + (V_1 - aV_2) Y_s \tag{3.9}$$

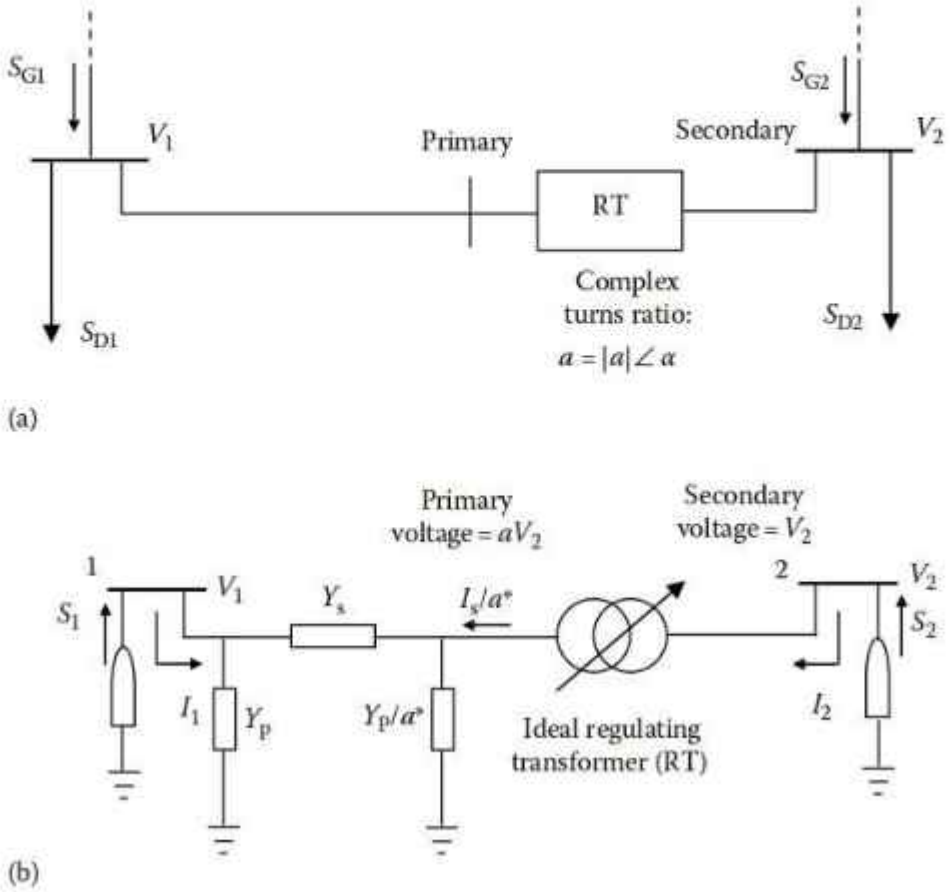


FIGURE 3.2
Two-bus example system demonstrating the use of the RT.

and

$$\frac{I_2}{a^*} = \frac{S_2^*}{(aV_2)^*} = aV_2Y_p + (aV_2 - V_1)Y_s.$$

We rewrite Equation 3.9 and obtain the pair

$$\begin{aligned} I_1 &= (Y_p + Y_s)V_1 + (-aY_s)V_2 \\ I_2 &= (-a^*Y_s)V_1 + aa^*(Y_p + Y_s)V_2. \end{aligned} \tag{3.10}$$

As a result, we conclude that the addition of RT changes the $\mathbf{Y}_{bus}$ to a new value

$$\mathbf{Y}_{bus} = \begin{bmatrix} Y_p + Y_s & -aY_s \\ -a^*Y_s & aa^*(Y_p + Y_s) \end{bmatrix}. \tag{3.11}$$

The addition of the RT has added the two additional variables $|a|$ and α to the previous ones and the linearized power balance equations will contain the

added difference variables $\Delta|a|$ and $\Delta\alpha$. In a practical situation, the RT is designed for either voltage magnitude or phase control. In the former case $\alpha = 0$ and $|a|$ can be changed in discrete steps $\Delta|a|$. In the latter case $|a|$ is constant and α can be changed in steps $\Delta\alpha$.

3.4 Practical Modeling Considerations

The power system model presented earlier is a simplified version of the actual state of the operational network. There are some additional complexities that deserve further consideration and more detailed modeling.

3.4.1 Generation Subsystem

The generator bus model presented thus far is a P–V model in which the turbine controls hold the generated active power at a fixed level and the machine voltage regulator maintains the bus voltage magnitude fixed. This model may be unrealistic if the voltage regulator is unable to set the field current to produce the desired bus voltage. Figure 3.3 shows a simplified circuit diagram of the generator. To increase $|V_t|$, the field current is increased such that $|E_f|$ increases. Under constant generated power conditions, the torque angle δ decreases slightly, but the phase of the armature current I_a will rotate such that the power factor angle ϕ increases. Thus more reactive power is injected into the bus. The limit on the control of $|V_t|$ occurs primarily due to the limit on the field current (and hence the limit on Q). In most cases, generators inject Q into the system since most loads have a lagging power factor. Thus, the salient limitation is related to the upper field current limit. In cases where the generator absorbs reactive power, the field current must be decreased. The limitations in this region are twofold: limits imposed by the maximum value of $|I_a|$ and practical stability limits associated with low field currents. In the latter case the dominant limitation may depend on the circuit power factor which always depends on the generator control designs. In addition, the generator complex volt–ampere rating $|S|$ warrants further

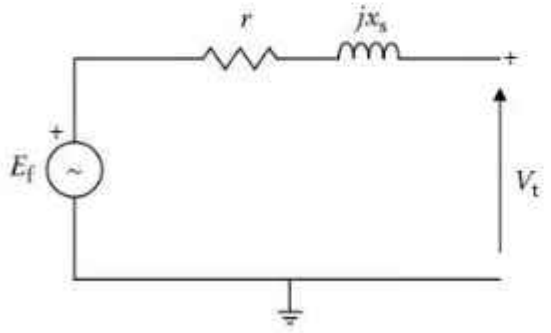


FIGURE 3.3
Circuit diagram for a synchronous generator.

attention. The generator $|S|$ rating implies possible further limitation of Q depending on the active power setting.

A simplified approach to generator field controller limits is usually followed for PF studies: a single upper limit on the injected Q for each generator is imposed. Under the assumption of an ideal P–V bus, the reactive power mismatch is not calculated since there is no specified reactive power at a generator bus. However, the Q generated by the machine must equal the total line flow reactive powers,

$$Q_{\text{generated at } i} = \sum_{j=1}^{N} |Y_{ij}| v_j v_i \sin(-\theta_{ij} + \delta_i - \delta_j).$$

To check whether the generated Q is below the upper limit,

$$Q_{\text{generated at } i} < Q^u_{\text{limit } i}$$

must hold. Similarly, the lower limit ($Q^l_{\text{limit } i} < 0$) must be checked

$$Q^l_{\text{limit } i} < Q_{\text{generated at } i} < Q^u_{\text{limit } i}. \tag{3.12}$$

If Equation 3.12 is violated at the upper or lower limit, the P–V bus model must be modified to reflect the physical reality. This usually entails conversion of the P–V bus to a P–Q bus where the specified Q is set at $Q^l_{\text{limit } i}$ or $Q^u_{\text{limit } i}$ as appropriate. The origin of Q limits of machines comes from the several operating conditions cited earlier, including limits on armature current, limits on field current and voltage, and operational stability requirements.

The concept and treatment of the slack bus requires further discussion. The PF formulation precludes specifying active and reactive power at all buses since the power lost in the transmission network is unknown. A formulation in which P is specified at all buses is overspecified and will yield results inconsistent with transmission losses. This is a mathematical consideration that translates to an operating condition: one machine in the system is usually operated such that the active power generated is set to hold the area error active power at zero. The term area error refers to the generation error in a given system, which causes a frequency error. If the net generation in a given system (an area) is greater than the load plus losses, the excess power injected will result in a net injected energy. This excess energy causes the machines in the system to accelerate. In other words, the integral of the error power (i.e., the error energy) becomes rotating kinetic energy. Similarly, if the net generation is too low, the system kinetic energy (and, hence, frequency) will decrease. The operational turbine setting is obtained at the (slack) swing machine from the area error: a raise power signal is required when the system frequency falls below the standard and a low power signal is required when the system frequency is too high.

The concept of a generalized swing machine was introduced in order to distribute area power error over several machines. Thus, the swing power is made up not at a single machine, but at several generators. Usually, a production computer program still uses a single bus as a swing bus for load-flow studies. One may encounter several swing buses in one load-flow study involving more than one area. (For purposes of this discussion, the term "area" refers to a coordinated power transmission network and generation in which a single swing bus is present.) There are several techniques for considering multiple swing buses, perhaps the simplest being separate load flows in which intertie buses are separated. It is possible to model the swing power in a distributed fashion (i.e., at several generators) by employing several swing machines.

To incorporate the transformer model into the PF formula we describe the various forms of the PF formulation, namely, the rectangular and the polar forms.

The matrix $\mathbf{Y}_{\text{bus}}$ represents the model of the passive portion of the n-bus network, in system form. From Equations 3.4 and 3.5 we conclude the following simple rules for finding the elements of the $\mathbf{Y}_{\text{bus}}$ matrix:

1. Diagonal element y_{ii} is obtained as the algebraic sum of all admittances incident to node i.
2. Off-diagonal elements $y_{ik} = y_{ki}$ are obtained as the negative of the admittance connecting nodes i and k.

It is usually the case that powers rather than currents are known. Then the PFEs have the nonlinear form

$$S_i^* = P_i - jQ_i = V_i^* \sum_{k=1}^{n} Y_{ik} V_k, \quad \text{for } i = 1, 2, \ldots, n. \tag{3.13}$$

This is the general form of the PFE. It should be noted that the n PFEs (Equation 3.13) are complex. These n equations therefore represent $2n$ real equations:

$$P_i = \sum_{k=1}^{n} |Y_{ik}||V_i||V_k| \cos(\delta_k - \delta_i + \gamma_{ik}) \triangleq f_{ip}, \quad \text{for } i = 1, 2, \ldots, n \tag{3.14}$$

$$Q_i = -\sum_{k=1}^{n} |Y_{ik}||V_i||V_k| \sin(\delta_k - \delta_i + \gamma_{ik}) \triangleq f_{iq}, \quad \text{for } i = 1, 2, \ldots, n. \tag{3.15}$$

Equations 3.14 and 3.15 express the balance of real and reactive powers at bus i, respectively. The nodal current can be written in the power form:

$$P_i - jQ_i = \left(V_i^*\right) \sum_{j=1}^{n} (Y_{ij} V_j). \tag{3.16}$$

Equation 3.16 can be expanded into various forms. Below are the commonly encountered rectangular and polar formulations.

3.4.1.1 Rectangular Formulation

The bus voltages on the right-hand side can be substituted by using the rectangular form $V_i = e_i + jf_i$. Choose the rectangular form; then we have by substitution

$$P_i = e_i\left(\sum_{j=1}^{n}(G_{ij}e_j - B_{ij}f_j)\right) + f_i\left(\sum_{j=1}^{n}(G_{ij}f_j - B_{ij}e_j)\right) \tag{3.17}$$

$$Q_i = f_i\left(\sum_{j=1}^{n}(G_{ij}e_j - B_{ij}f_j)\right) - e_i\left(\sum_{j=1}^{n}(G_{ij}f_j + B_{ij}e_j)\right), \tag{3.18}$$

where the admittance is expressed in the rectangular form

$$Y_{ij} = G_{ij} + jB_{ij}. \tag{3.19}$$

3.4.1.2 Polar Formulation

Using the notation

$$\begin{aligned} V_i &= |V_i|e_i^{j0} \\ &= |V_i|\angle\theta_i, \end{aligned} \tag{3.20}$$

we obtain the following representation

$$P_i = |V_i|\sum_{j=1}^{n}|Y_{ij}||V_j|\cos(\theta_i - \theta_j - \psi_{ij}) \tag{3.21}$$

$$Q_i = |V_i|\sum_{j=1}^{n}|Y_{ij}||V_j|\sin(\theta_i - \theta_j - \psi_{ij}), \tag{3.22}$$

where the admittance is expressed in the polar form: $Y_{ij} = |Y_{ij}|\psi_{ij}$.

3.5 Iterative Techniques for PF Solution

PFEs are large-scale nonlinear sets of equations that require using iterative techniques to obtain their solution. The following major methods are discussed: Gauss–Seidel (G–S) iterative technique, Newton–Raphson method (N–R), and fast-decoupled Newton method.

3.5.1 G–S Iterative Technique

This is a simple iterative method that was very popular in the early days of digital computer-based PF analysis. The more powerful N–R method dominates the field today. The G–S method is still used for small power systems where program simplicity is more important than computing costs, and, in many cases, it is used in large-scale systems to obtain an "initial solution" for the N–R program [2].

3.5.1.1 G–S Algorithm

We consider solving an n-dimensional equation of the type

$$F(x) = 0. \tag{3.23}$$

We introduce the method by first solving the following scalar equation iteratively.

$$f(x) = 0. \tag{3.24}$$

We need to reshape the given function $f(x)$ into the alternative form

$$x = F(x). \tag{3.25}$$

For a given function f, it is always possible to find a function F. As an example, consider the second-order equation

$$f(x) = x^2 - 2x + 5 = 0.$$

The function $f(x)$ represents a parabola. We can obviously write $f(x)$ as

$$x = \underbrace{\frac{1}{2}x^2 + \frac{5}{2}}_{F(x)},$$

which is of the form of Equation 3.23.

Note that $F(x)$ is not unique. In the present example we could also have written

$$x = \underbrace{\frac{2x+5}{x}}_{F(x)}.$$

The algorithm is based on the following empirical reasoning.

The equations $x - F(x) = 0$ and $f(x) = 0$ must have identical roots. The function $x - F(x)$ represents a "boxed in" region between the sloping line x

and the function $F(x)$. As this region "thins out" to zero close to a root, that root can be reached by a zigzag search process.

3.5.1.2 G–S Method Applied to the PFEs

3.5.1.2.1 Load Buses

For this case, real power P and reactive power Q are given. We assume an n-node system including the slack bus S where both V and ϕ are specified and remain fixed throughout. Since $|P|$ and $|Q|$ are given for all buses except the slack bus, we have for any bus K,

$$I_K = \frac{P_K - jQ_K}{V_K^*}, \quad \text{for} \begin{cases} K = 1, 2, \ldots, n \\ K \neq S \end{cases}, \tag{3.26}$$

where S is the slack bus.

Now the performance equation in the bus frame of reference using Y_{bus} where ground is included as a reference node will be

$$I_{\text{BUS}} = [Y_{\text{BUS}}]\overline{V}_{\text{BUS}}. \tag{3.27}$$

For an n-node system, there are $n-1$ linear independent equations to solve. Expanding Equation 3.27, we get

$$I_1 = Y_{11}V_1 + Y_{12}V_2 + \cdots$$

and hence for Kth bus

$$I_K = Y_{K1}V_1 + Y_{K2}V_2 + \cdots = \sum_{q=1}^{n} Y_{Kq}V_q; \tag{3.28}$$

that is,

$$\begin{aligned} I_K &= \sum_{q=1}^{n} Y_{Kq}V_q, \quad K = 1, \ldots, n; \quad K \neq S \\ &= Y_{KK}V_K + \sum_{\substack{q=1 \\ q \neq K}}^{n} Y_{Kq}V_q. \end{aligned} \tag{3.29}$$

Thus

$$V_K = \frac{1}{Y_{KK}}\left[I_K - \sum_{\substack{q=1 \\ q \neq K}}^{n} Y_{Kq}V_q\right], \quad \text{for } K = 1, \ldots, n; \quad K \neq S.$$

Substituting for I_K from Equation 3.26, we have

$$V_K = \frac{1}{Y_{KK}}\left[\frac{P_K - jQ_K}{V_k^*} - \sum_{\substack{q=1 \\ q \neq K}}^{n} Y_{Kq}V_q\right], \tag{3.30}$$

where

$K = 1, \ldots, n$

$K \neq S$

In the direct Gauss method, we assume guess values for voltage for all buses except the slack bus where the voltage magnitude and phase angle are specified and remain fixed. Normally, we set the voltage magnitude and phase angle of these buses to be equal to that of the slack bus and work in the per unit system (i.e., we may take in pu, the voltage magnitude and phase angle as $1\angle 0$). The assumed bus voltage and the slack bus along with P and Q are substituted in the right-hand side of Equation 3.30 to obtain a new and improved set of bus voltages. After the entire iteration is completed, the new set of bus voltages is again substituted along with the specified slack bus voltage in the right-hand side of Equation 3.30 to obtain a new set of bus voltages. The process is continued until

$$|V_i^{k+1} - V_i^k| \leq \varepsilon, \tag{3.31}$$

where

k is an iteration count

ε is a very small number, which depends upon the system accuracy and is normally equal to 10^{-3} in the per unit system

The iterations are continued until the node of the bus voltage obtained at the current iteration minus the value of the bus voltage at the previous iteration is less than a chosen very small number and in this way we obtain the solution for $|V|$ and ϕ.

3.5.1.3 G–S Iterative Technique

In the G–S method, the value of a bus voltage calculated for any bus immediately replaces the previous values in the next step. This contrasts the case of the Gauss method where the calculated bus voltages replace the earlier value only at the end of the complete iteration. Normally the G–S method converges much faster than the Gauss method; that is, the number of iterations needed to obtain the solution is much less in the G–S method, as compared to the Gauss method.

3.5.1.3.1 Generator Bus

This is the voltage-controlled bus where $|P|$ and $|V|$ are specified. However, usually there is a limit on reactive power; that is, $Q_{\max}$ and $Q_{\min}$ to hold the generation voltage within limits are also given. For any bus p we have

$$P_p - jQ_p = V_p^* I_p \tag{3.32}$$

and

$$I_p = \sum_{q=1}^{n} Y_{pq} V_q. \tag{3.33}$$

Then substituting Equation 3.33 into Equation 3.32, we get

$$P_p - jQ_p I_p = V_p^* \sum_{q=1}^{n} Y_{pq} V_q. \tag{3.34}$$

As a result

$$Q_p = \text{Imaginary}\left\{ V_p^* \sum_{q=1}^{n} Y_{pq} V_q \right\}. \tag{3.35}$$

The iterative process starts by assuming the values of bus voltages and the phase angles, say $|V|$ and ϕ, from which we determine the real and imaginary components (e.g., e and f) of the bus voltages.

$$\begin{aligned} e_p &= V_p \cos \phi_p \\ f_p &= V_p \sin \phi_p. \end{aligned} \tag{3.36}$$

For any bus p, V_p is specified and, therefore, the selected value of e_p and f_p must satisfy Equation 3.37 approximately.

$$e_p^2 + f_p^2 \simeq [|V_p| \text{ scheduled}]^2, \tag{3.37}$$

where V_p (scheduled) is the specified bus voltage for any bus p.

We now substitute the assumed values of e_p and f_p in Equation 3.35 to calculate the reactive power Q. If the calculated reactive power exceeds $Q_{\max}$ or is below $Q_{\min}$, then we do the following:

$$\text{If } Q > Q_{\max}, \text{ set } Q = Q_{\max} \quad \text{or} \quad \text{if } Q < Q_{\min}, \text{ set } Q = Q_{\min}, \tag{3.38}$$

and treat this bus as a load bus to find the voltage solution. If Equation 3.38 is not true, then use the phase angle of this assumed bus voltage (i.e., ϕ_p) to recalculate e_p and f_p for any bus p.

We know that $\phi_p^k = \tan^{-1}\left(f_p^k / e_p^k\right)$, where k is the iteration count. Assuming this phase angle ϕ_p also to be that of the scheduled bus voltage V_p (for this bus, ϕ_p is unknown but V_p is given), we get

$$\begin{aligned} e_p^k \text{ (new)} &= |V_p \text{ scheduled}| \cos \phi_p^k \\ f_p^k \text{ (new)} &= |V_p \text{ scheduled}| \sin \phi_p^k. \end{aligned} \tag{3.39}$$

We substitute the e_p^k and f_p^k from Equation 3.39 in Equation 3.30 to recalculate bus voltages and the process is continued until the process converges. However, at every step the reactive power is also calculated to check whether the calculated Q is within the limit.

3.5.1.3.2 *Acceleration Factor*

We use acceleration factors to increase the rate of convergence. The choice of an acceleration factor depends upon the system and appropriate values normally lie within 1.4–1.6. Then, after calculating e_p^{k+1} and f_p^{k+1} at the $k+1$st iteration and knowing the acceleration factor, say α and β, we calculate the new estimate for the bus voltages given as

$$\begin{aligned} e_p^{k+1}\,(\text{accelerated}) &= e_p^k + \alpha\left[e_p^{k+1} - e_p^k\right] \\ f_p^{k+1}\,(\text{accelerated}) &= f_p^k + \beta\left[f_p^{k+1} - f_p^k\right] \end{aligned} \tag{3.40}$$

and this new estimate replaces the calculated values e_p^{k+1} and f_p^{k+1}.

3.5.1.4 Line Flow and Losses

After calculating bus voltages and their phase angles for all the buses, the line flow and line losses are calculated. We assume the normal π representation of the transmission line, as shown in Figure 3.4. Having found the solution of bus voltages (both $|V|$ and ϕ for load buses and only ϕ for a generation bus), we calculate the line flows between any buses p and q.

Let i_{pq} be the current flow from bus p toward q; then

$$i_{pq} = [V_p - V_p]Y_{pq} + V_p\frac{Y'_{pq}}{2}, \tag{3.41}$$

where V_p and V_q are the bus voltages at the buses p and q that have been calculated from the load-flow studies.

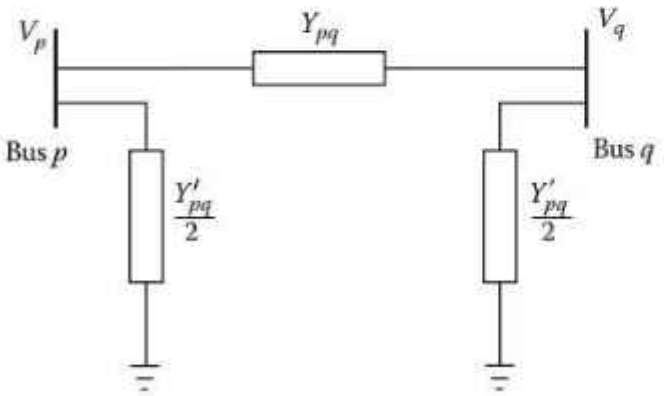


FIGURE 3.4
π-representation of transmission line.

The PF in the line p–q at the bus p is given by

$$\begin{aligned} p_{pq} - jQ_{pq} &= V_p^* i_{pq} \\ &= V_p^*[V_p - V_q]Y_{pq} + V_p^* V_p \frac{Y'_{pq}}{2}. \end{aligned} \tag{3.42}$$

Similarly the line flow in the line p–q at bus q is given by

$$p_{pq} - jQ_{pq} = V_q^*[V_q - V_p]Y_{pq} + V_q^* V_q \frac{Y'_{pq}}{2}. \tag{3.43}$$

The algebraic sum of Equations 3.42 and 3.43 is the line losses in the element p–q.

3.5.2 N–R Method

The G–S iterative algorithm (Figure 3.5) is very simple but convergence becomes increasingly slow as the system size grows. The N–R technique (Figure 3.6) converges fast in less than 4–5 iterations regardless of system size. It is popular for large system studies.

3.5.2.1 N–R Algorithm in the Scalar Case

In this method, we start with the scalar Equation 3.21. Assume an initial solution $x^{(0)}$. We try to evaluate the error $\Delta x^{(0)}$ associated with our guess. As $x^{(0)} + \Delta x^{(0)}$ by definition is the correct root then we must require that

$$f(x^{(0)}) + \Delta x^{(0)} = 0. \tag{3.44}$$

If we expand $f(x)$ in a Taylor series around the initial guess value, we obtain

$$f(x^{(0)}) + \Delta x^{(0)} \left(\frac{df}{dx}\right)^{(0)} + \frac{1}{2}(\Delta x^{(0)})^2 \left(\frac{d^2 f}{d^2 x}\right)^{(0)} + \cdots = 0. \tag{3.45}$$

All derivatives are computed for $x = x^{(0)}$.

Assuming that the error is relatively small, then the higher-order terms can be neglected and we have

$$f(x^{(0)}) + \Delta x^{(0)} \left(\frac{df}{dx}\right)^{(0)} \approx 0, \tag{3.46}$$

from which we then compute an approximate value for the error

$$\Delta x^{(0)} = -\frac{f(x^{(0)})}{(df/dx)^{(0)}}. \tag{3.47}$$

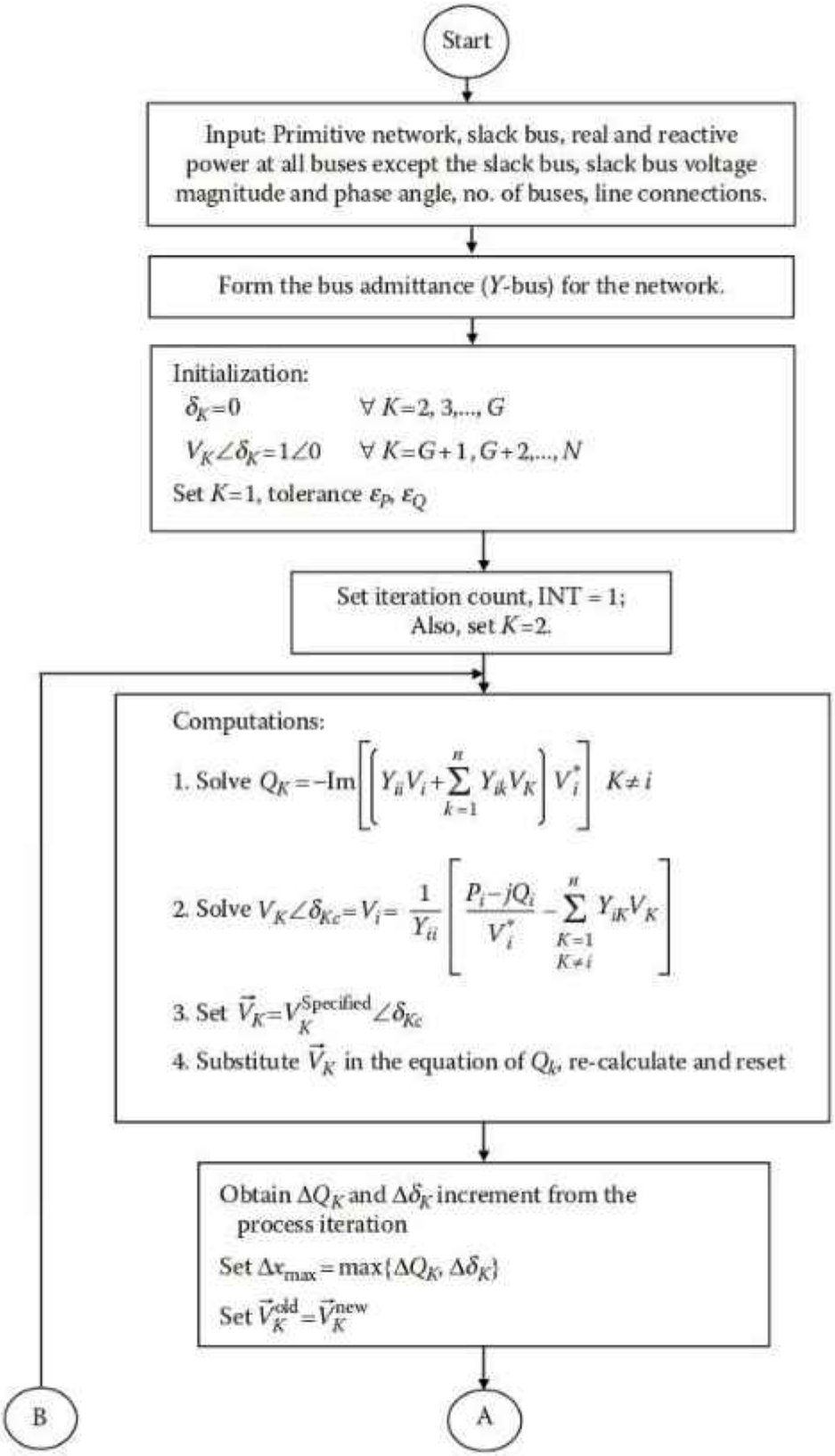


FIGURE 3.5
G–S method.

(continued)

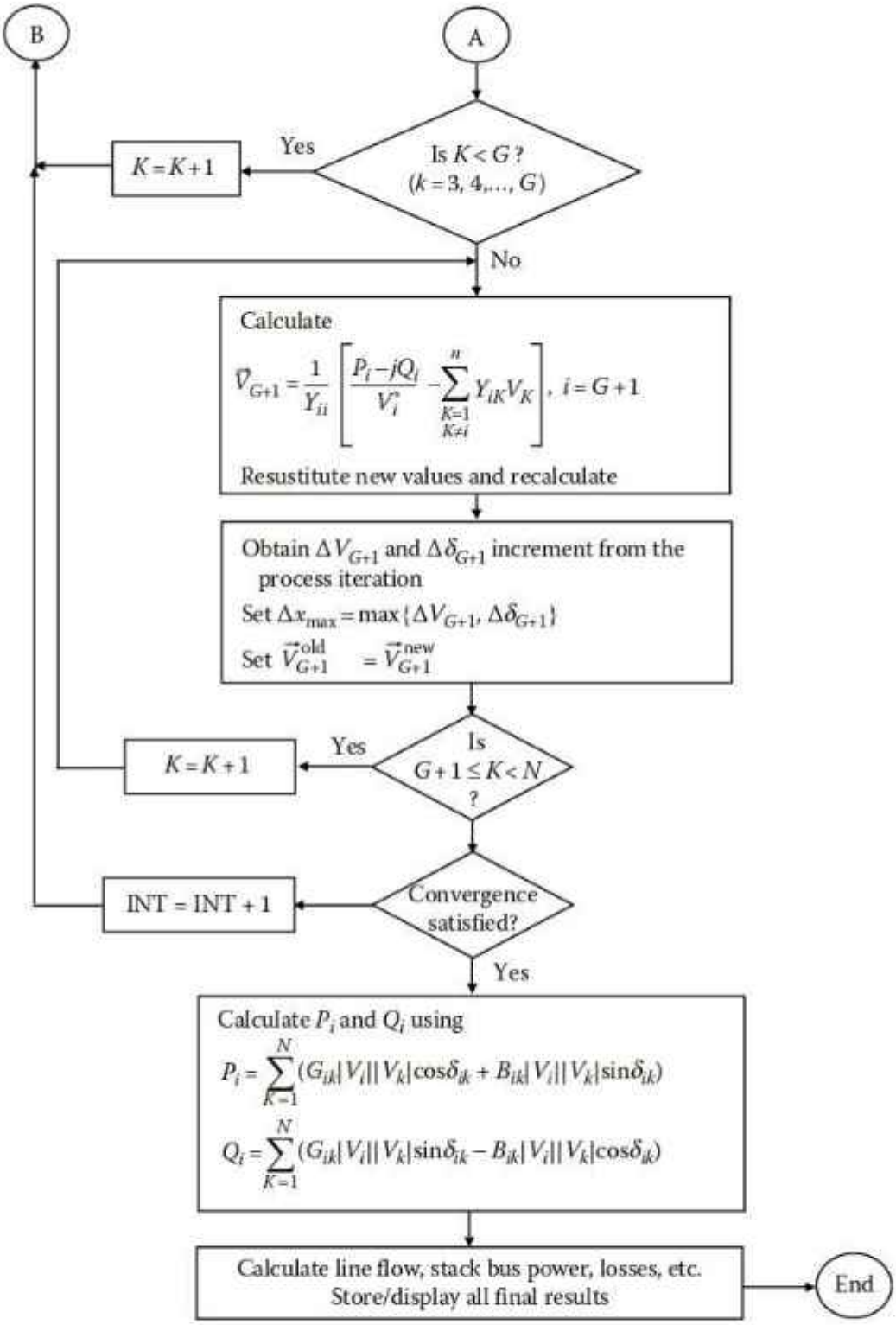


FIGURE 3.5 (continued)

We add this error to the original guess and should thus obtain an improved value $x^{(1)}$ for the root, that is,

$$x^{(1)} = x^{(0)} + \Delta x^{(0)} = x^{(0)} - \frac{f(x^{(0)})}{(df/dx)^{(0)}}.$$

Repeated use of this procedure thus yields the N–R algorithm:

$$x^{(v+1)} = x^{(v)} - \frac{f(x^{(v)})}{(df/dx)^{(v)}}. \tag{3.48}$$

We illustrate the method by finding the larger root of the quadratic equation in x: $f(x) = x^2 - 5x + 4$. Use the initial guess $x^{(0)} = 6$.

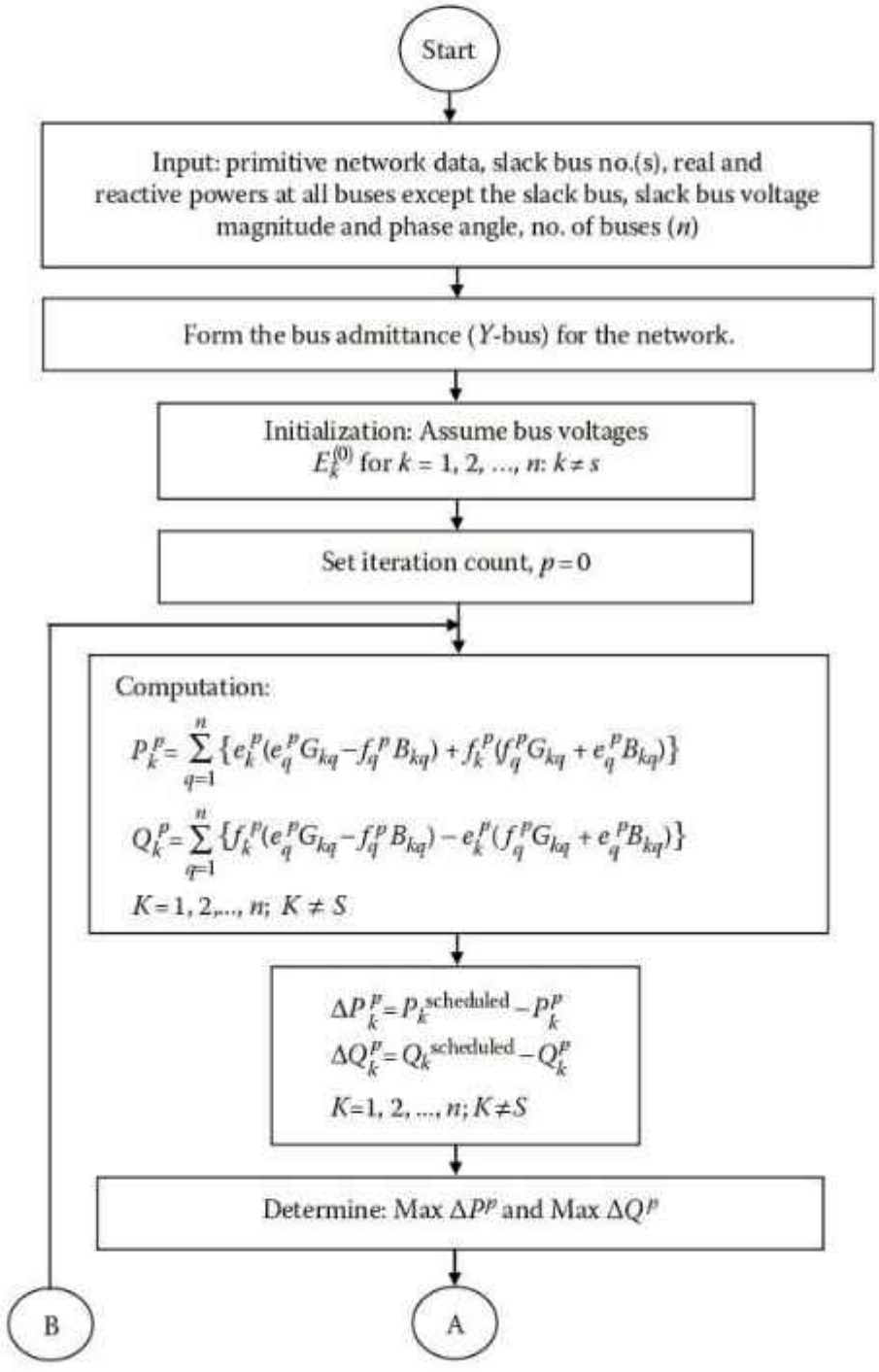


FIGURE 3.6
N–R method.

(*continued*)

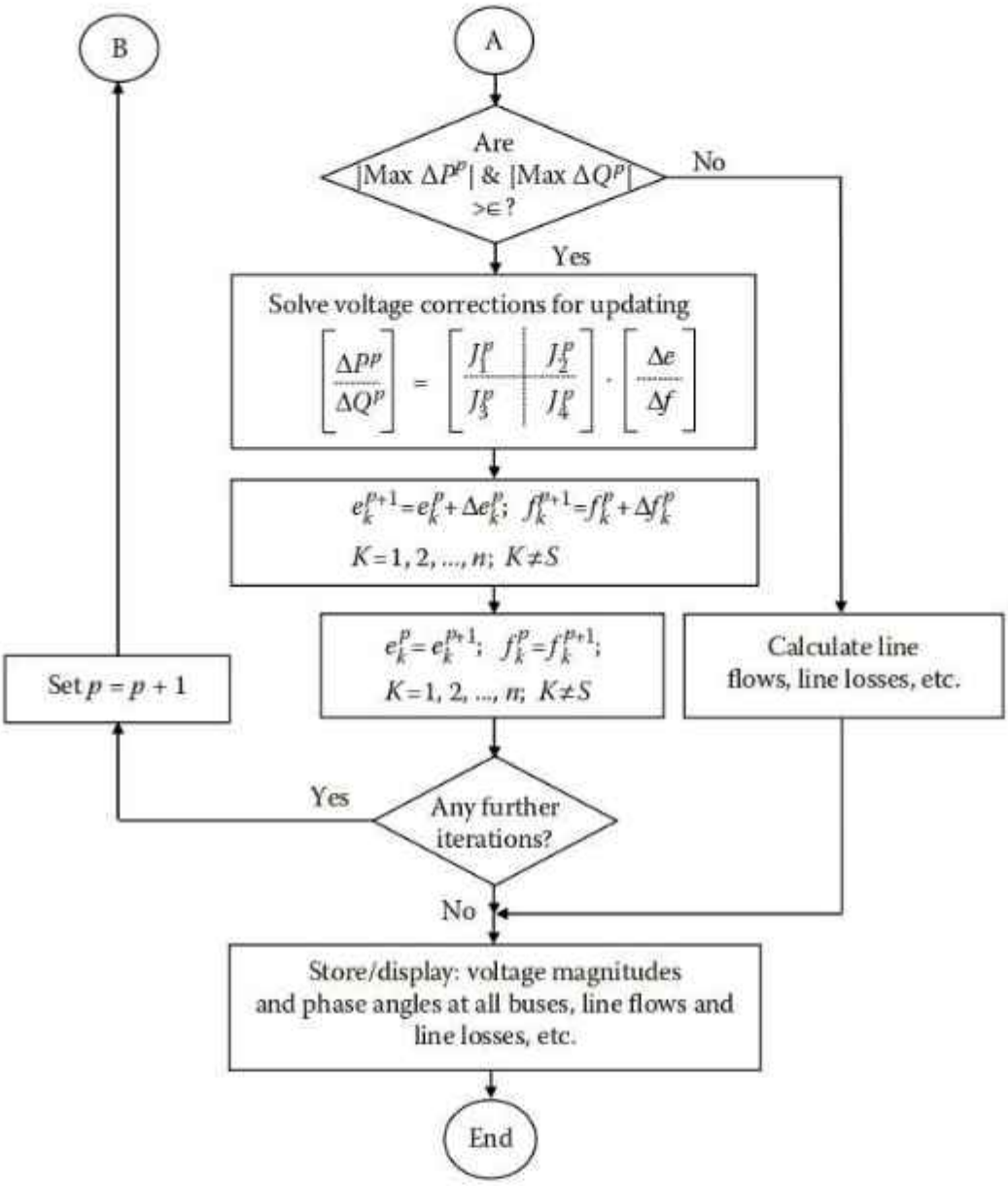


FIGURE 3.6 (continued)

We have

$$\frac{df}{dx} = 2x - 5.$$

The computations proceed according to the following steps:

1. $f(x^{(0)}) = f(6) = 6^2 - 5 \times 6 + 4 = 10.$
2. $\left(\frac{df}{dx}\right)^{(0)} = 2 \times 6 - 5 = 7.$
3. $\Delta x^{(0)} = -10/7 = -1.429.$
4. $x^{(1)} = 6 - 1.429 = 4.571.$

Two additional iterations yield

$$x^{(2)} = 4.079$$
$$x^{(3)} = 4.002.$$

The correct value is 4 and we have thus arrived within 0.05% in only three iterations. As in the case of the G–S algorithm, one does not have a full convergence guarantee, unless the initial point is suitably chosen. This, of course, is more important when attempting to solve large-scale, nonlinear or multidimensional system of equations.

3.5.2.2 *N–R Algorithm in the n-Dimensional Case*

Consider now the solution of the n-dimensional Equation 3.21. By expanding each equation in a Taylor series around the initial guess

$$x^{(0)} = \begin{bmatrix} x_1^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix} \tag{3.49}$$

and upon retaining only the first derivative terms we have

$$\begin{aligned} &f_1(x^{(0)}) + \left(\frac{\partial f_1}{\partial x_1}\right)^{(0)} \Delta x_1 + \cdots + \left(\frac{\partial f_1}{\partial x_n}\right)^{(0)} \Delta x_n \approx 0 \\ &\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ &f_n(x^{(0)}) + \left(\frac{\partial f_n}{\partial x_1}\right)^{(0)} \Delta x_1 + \cdots + \left(\frac{\partial f_n}{\partial x_n}\right)^{(0)} \Delta x_n \approx 0. \end{aligned} \tag{3.50}$$

We can write this system of n linear equations as follows:

$$\begin{bmatrix} f_1(x^{(0)}) \\ \vdots \\ f_n(x^{(0)}) \end{bmatrix} + \begin{bmatrix} \left(\frac{\partial f_1}{\partial x_1}\right)^{(0)} & \cdots & \left(\frac{\partial f_1}{\partial x_n}\right)^{(0)} \\ \vdots & & \vdots \\ \left(\frac{\partial f_n}{\partial x_1}\right)^{(0)} & \cdots & \left(\frac{\partial f_n}{\partial x_n}\right)^{(0)} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix} \approx \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Or, in compact matrix–vector notation,

$$\mathbf{f}(\mathbf{x}^{(0)}) + \mathbf{J}^{(0)} \Delta \mathbf{x}^{(0)} \approx 0. \tag{3.51}$$

From this last equation, we then solve for the error vector

$$\Delta \mathbf{x}^{(0)} \approx -[\mathbf{J}^{(0)}]^{-1} \mathbf{f}(\mathbf{x}^{(0)}). \tag{3.52}$$

Upon adding this vector to the original guess and at the same time indicating a repetitive process we obtain the N–R algorithm

$$\mathbf{x}^{(v+1)} = \mathbf{x}^{(v)} - [\mathbf{J}^{(v)}]^{-1}\mathbf{f}(\mathbf{x}^{(v)}). \tag{3.53}$$

The $n \times n$ matrix **J**, having as its elements the partial derivatives $\partial f_i/\partial x_j$, is referred to as a Jacobian. Use of this algorithm obviously necessitates the need of inverting an $n \times n$ matrix in each iteration.

3.5.2.3 N–R Algorithm Applied to the PFEs

The solution of the power system problem of a nonlinear set of power equations is popularly accomplished by using the N–R technique, which is applicable to the rectangular and polar formulations of the problem. The two cases are presented below.

3.5.2.3.1 Case A: Formulation of the N–R Method in Rectangular Form

Based on the help of the N–R iterations, the general nonlinear algebraic equations of power are transformed into a set of linear algebraic equations relating the changes in power (i.e., error in power) to the change in real and reactive components of bus voltages with the help of the Jacobian matrix. This is actually done by expanding the function by a Taylor's series and neglecting higher-order derivatives and higher power. It is necessary that the initial guess be close to the solution point.

$$\begin{bmatrix} \Delta P_1 \\ \cdots \\ \Delta P_{n-1} \\ \cdots \\ \Delta Q_1 \\ \cdots \\ \Delta Q_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{\partial P_1}{\partial e_1} & \frac{\partial P_1}{\partial e_{n-1}} & \frac{\partial P_1}{\partial f_1} & \frac{\partial P_1}{\partial f_{n-1}} \\ \frac{\partial P_{n-1}}{\partial e_1} & \frac{\partial P_{n-1}}{\partial e_{n-1}} & \frac{\partial P_{n-1}}{\partial f_1} & \frac{\partial P_{n-1}}{\partial f_{n-1}} \\ \frac{\partial Q_1}{\partial e_1} & \frac{\partial Q_1}{\partial e_{n-1}} & \frac{\partial Q_1}{\partial f_1} & \frac{\partial Q_1}{\partial f_{n-1}} \\ \frac{\partial Q_{n-1}}{\partial e_1} & \frac{\partial Q_{n-1}}{\partial e_{n-1}} & \frac{\partial Q_{n-1}}{\partial f_1} & \frac{\partial Q_{n-1}}{\partial f_{n-1}} \end{bmatrix} \begin{bmatrix} \Delta e_1 \\ \vdots \\ \Delta e_{n-1} \\ \vdots \\ \Delta f_1 \\ \vdots \\ \Delta f_{n-1} \end{bmatrix} \tag{3.54}$$

where the nth bus is the slack bus. Equation 3.54 can be expressed as

$$\begin{bmatrix} \Delta P \\ \cdots \\ \Delta Q \end{bmatrix} = \begin{pmatrix} J_1 & \vdots & J_2 \\ \cdots & \cdots & \cdots \\ J_3 & \vdots & J_4 \end{pmatrix} \begin{bmatrix} \Delta e \\ \cdots \\ \Delta f \end{bmatrix}, \tag{3.55}$$

where J_1, J_2, J_3, and J_4 are the elements of the Jacobian matrix that are calculated from the expression of power as follows.

Elements of J_1
Off-diagonal elements

$$\partial P_p/\partial e_q = e_p G_{pq} - f_p B_{pq}, \quad \text{for } q = p. \tag{3.56}$$

Diagonal elements

$$\begin{aligned}\frac{\partial P_p}{\partial e_p} &= 2e_p G_{pp} - f_p B_{pp} - f_p B_{pp} \\ &= \sum_{\substack{q=1 \\ q \neq p}}^{n} [e_q G_{pq} - f_q B_{pq}].\end{aligned} \tag{3.57}$$

Elements of J_2
Off-diagonal elements

$$\partial P_p/\partial f_q = e_p B_{pq} + f_p G_{pq} \quad q \neq p. \tag{3.58}$$

Diagonal elements

$$\begin{aligned}\partial P_p/\partial f_p = e_p B_{pp} + 2f_p G_{pp} - e_p B_{pp} \\ + \sum_{\substack{q=1 \\ q \neq p}}^{n} [f_q G_{pq} - e_q B_{pq}].\end{aligned} \tag{3.59}$$

Elements of J_3
Off-diagonal elements

$$\partial Q_p/\partial e_q = f_p G_{pq} + e_p B_{pq}, \quad \text{for } q \neq p. \tag{3.60}$$

Diagonal elements

$$\partial Q_p/\partial e_p = f_p G_{pp} - f_p G_{pp} + 2e_p B_{pp} - \sum_{\substack{q=1 \\ q \neq p}}^{n} [f_p G_{pq} - e_q B_{pq}]. \tag{3.61}$$

Elements of J_4
Off-diagonal elements

$$\partial Q_p/\partial f_q = f_p B_{pq} - e_p B_{pq}, \quad \text{for } q \neq p. \tag{3.62}$$

Diagonal elements

$$\frac{\partial Q_p}{\partial f_p} = e_p G_{pp} + 2f_p B_{pp} - e_p G_{pp} + \sum_{\substack{q=1 \\ q \neq p}}^{n} [e_q G_{pq} - f_q B_{pq}]. \tag{3.63}$$

The following steps are performed in sequence to obtain the PF solution by N–R method:

1. For load buses where P and Q are given, we assume the bus voltages' magnitude and phase angle for all the buses except the slack bus where $|V|$ and ϕ are specified. Normally, we use a flat voltage start; that is, we set the assumed bus voltage magnitude and its phase angle (in other words the real and imaginary component e and f of the bus voltages) equal to the slack bus values.
2. Substituting these assumed bus voltages (i.e., e and f), we calculate the real and reactive components of power (i.e., P_p and Q_p) for all the buses $p = 1, \ldots, n-1$ except the slack bus.
3. Since P_p and Q_p for any bus p are given, the error in the power will be

$$\begin{aligned} \Delta P_p^k &= P_p \text{ (scheduled)} - P_p^k \\ \Delta Q_p^k &= Q_p \text{ (scheduled)} - Q_p^k, \end{aligned} \tag{3.64}$$

   where
   k is an iteration count
   P_p^k and Q_p^k are the active and reactive power values calculated with the latest value of bus voltages at any iteration k

4. Elements of the Jacobian matrix (J_1, J_2, J_3, and J_4) are calculated with the latest bus voltages and calculated power equations.
5. We then solve the linear set of Equations 3.54 by either an iterative technique or by the method of elimination (normally by the Gaussian elimination method) to determine the voltage correction, that is Δe_p and Δf_p at any bus p.
6. This value of voltage correction is used to determine the new estimate of bus voltages as follows:

$$\begin{aligned} e_p^{k+1} &= e_p^k + \Delta e_p^k \\ f_p^{k+1} &= f_p^k + \Delta f_p^k, \end{aligned}$$

   where k is an iteration count.
7. Now this new estimate of the bus voltage e_p^{k+1} and f_p^{k+1} is used to recalculate the error in power and, thus, the entire algorithm starting from step 3 as given earlier is repeated.

During each iteration, the elements of the Jacobian are calculated since they depend upon the latest voltage estimate and calculated power. The process is continued until the error in power becomes very small, that is,

$$|\Delta P| < \varepsilon \quad \text{and} \quad |\Delta Q| < \varepsilon, \tag{3.65}$$

where ε is a very small number. This method converges faster than the G–S method and exhibits quadratic convergence properties. Moreover, while the number of iterations to obtain the nominal solutions increases with the problem size in the case of the G–S method, the number of iterations to obtain a converged solution is nearly constant in the N–R method. Later, we show that with the modified Newton method, nearly 5–6 iterations are needed to obtain a solution although the time taken to complete an iteration is nearly seven times as that of the G–S method.

Example 3.5.1

Solve the following two equations in x_1 and x_2 using the N–R method:

$$F_1(x) = x_1^2 + x_2^2 - 4x_1 = 0$$
$$F_2(x) = x_1^2 + x_2^2 - 8x_1 + 12 = 0.$$

1. Find expressions for the elements of the Jacobian matrix and find the correction increments Δx_1, Δx_2.
2. Calculate the first five iterations to find estimates of the solution using the following initial guesses:

(a) $x_1 = 2, \quad x_2 = 4.$
(b) $x_1 = -5, \quad x_2 = -5.$
(c) $x_1 = -0.1, \quad x_2 = 1.$

Solution

The Jacobian elements are as follows:

$$\frac{\partial F_1}{\partial x_1} = 2x_1 - 4; \quad \frac{\partial F_1}{\partial x_2} = 2x_2; \quad \frac{\partial F_2}{\partial x_1} = 2x_1 - 8; \quad \frac{\partial F_2}{\partial x_2} = 2x_2.$$

Now, using the formulation for the typical N–R problem, we have

$$\begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} -F_1(x) \\ -F_2(x) \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 - 4 & 2x_2 \\ 2x_1 - 8 & 2x_2 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} -(x_1^2 + x_2^2 - 4x_1) \\ -(x_1^2 + x_2^2 - 8x_1 + 12) \end{bmatrix}.$$

The solution for Δx_1, Δx_2 is given by

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - 4 & 2x_2 \\ 2x_1 - 8 & 2x_2 \end{bmatrix}^{-1} \begin{bmatrix} -(x_1^2 + x_2^2 - 4x_1) \\ -(x_1^2 + x_2^2 - 8x_1 + 12) \end{bmatrix}$$

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} 3 - x_1 \\ -(x_1^2 + x_2^2 - 8x_1 + 12) \quad 12x_2 \end{bmatrix}.$$

As a result the new estimates of the solution are given by

$$x_i^{n+1} = x_i^n + \Delta x_i^n$$
$$x_i^{n+1} = x_i^n + (3 - x_1^n)$$
$$x_i^{n+1} = 3.$$

Note that x_1 is independent of the starting point. In addition, the iterative formula for x_2 is

$$x_2^{(n+1)} = x_2^{(n)} + \frac{x_1^{2(n)} - x_2^{2(n)} - 6x_1^{(n)} + 12}{2x_2^n}$$
$$= \frac{x_1^{2n} + x_2^{2n} - 6x_1^n + 12}{2x_2^n},$$

with $x_i^n = 3$,

$$x_2^{(n+1)} = \frac{x_2^{2n} + 3}{2x_2^n}.$$

Tables 3.2 through 3.4 give different values of x_1 and x_2 at different iterations.

TABLE 3.2

Iteration Process with Start Point (2, 4)

Iteration	x_1	x_2
0	2	4
1	3	2.5
2	3	1.85
3	3	1.74
4	3	1.7321
5	3	1.73205

TABLE 3.3

Iteration Process with Start Point (−5, −5)

Iteration	x_1	x_2
0	−c5	−5
1	3.0	−9.2
2	3.0	−4.763
3	3.0	−2.6964
4	3.0	−1.9045
5	3.0	−1.7399
6	3.0	−1.7321
7	3.0	−1.73205

TABLE 3.4

Iteration Process with Start Point (−0, 1)

Iteration	x_1	x_2
0	−0.1	1
1	3.0	6.805
2	3.0	3.623
3	3.0	2.225
4	3.0	1.787
5	3.0	1.7329
6	3.0	1.73205

Example 3.5.2

Use the N–R method to find the roots of the equation

$$F(x) = x^3 - 6x^2 + 11x - 6.$$

Assume the following initial guesses: $x = 0$, 0.5, 1.5, 2.5, and 4.

SOLUTION

$$F(x) = x^3 - 6x^2 + 11x - 6$$
$$F'(x) = 3x^2 - 12x + 11$$
$$\Delta x = -\frac{F}{F'} = -\frac{x^3 + 6x^2 - 11x + 6}{3x^2 - 12x + 11}$$

New estimates are given by

$$x^{n+1} = x^n + \Delta x.$$

For the different initial points the solution is developed in Table 3.5. Note that the equation has roots at the points where $x = 1.00$, 2.00, and 3.00. We converged on the first and the third roots depending on the starting estimate. The proximity of the initial guess to the solution affects the N–R solutions of the problem.

TABLE 3.5

Iteration Process of Example 3.5.2 with Different Initial Points

Iteration	Value of the Variable x at the End of Each Iteration for the Five Initial Values Used				
0 (initial)	0.0000	0.5000	1.5000	2.5000	4.0000
1	0.5450	0.8260	3.0000	1.0000	3.4500
2	0.8499	0.9677	—	—	3.1500
3	0.9747	0.9985	—	—	3.0300
4	0.9909	1.0000	—	—	3.0009
5	1.0000	—	—	—	3.0000

Example 3.5.3

For the network shown in Figure 3.7, do the following:

1. Compute the Y-bus admittance matrix.
2. Write down the load-flow equations of the problem using the rectangular formulation.

The nodal admittances are given as

$$Y_{11} = 4 - j5, \quad Y_{22} = 4 - j10, \quad Y_{33} = 8 - j15$$
$$Y_{12} = 0, \quad Y_{13} = -4 + j5, \quad Y_{23} = -4 + j10.$$

We also have

$$|V_1| = 1, \quad \theta_1 = 0$$
$$P_2 = 1.7, \quad |V_2| = 1.1249$$
$$P_3 = -2, \quad Q_3 = -1.$$

Solution

The load-flow equations are obtained utilizing the following formula:

$$P_i - jQ_i = V_i^* \sum Y_{ij} V_j.$$

Recall Equation 3.23. To obtain the rectangular forms, we have for bus 1

$$P_1 - 1Q_1 = (1 + j0[(4 - j5)(1 + j0) + (-4 + j5)(e_3 + jf_3)]).$$

This yields

$$P_1 = 4 - e_3 - 5f_3.$$
$$-Q_1 = -5 + 5e_3 - 4f_3.$$

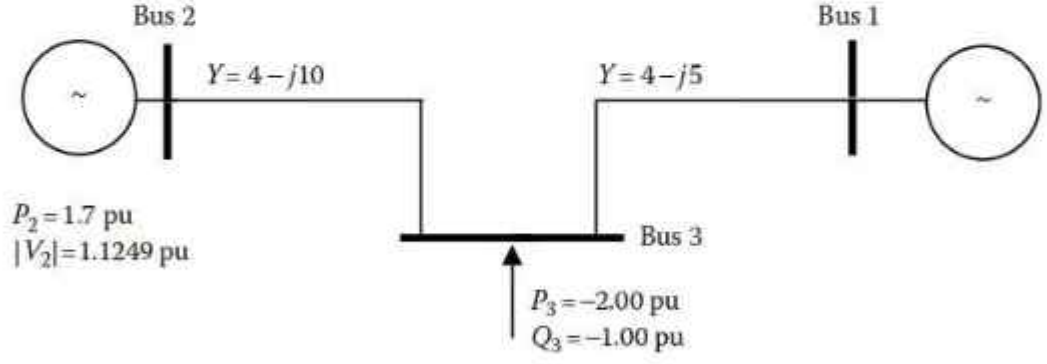


FIGURE 3.7
Single-line diagram for Example 3.5.3.

For bus 2 we have

$$1.7 - jQ_2 = (e_2 - jf_2)[(4 - j10)(e_2 + jf_2) + (-4 + j10)(e_3 + jf_3)].$$

This reduces to

$$1.7 - jQ_2 = (4 - j10)(e_2^2 + jf_2^2) + (e_2 - jf_2)(-4 + j10) * (e_3 + jf_3).$$

Using

$$|V_2|^2 = e_2^2 + f_2^2 = (1.1249)^2,$$

we obtain

$$1.7 - jQ_2 = 5.0616 - j12.654 + (-4 + j10)[(e_2e_3 + f_1f_3) + j(f_3e_2 - f_2e_3)].$$

Separating real and imaginary parts, we get

$$\begin{aligned} 1.7 &= 5.0616 - 4(e_2e_3 + f_2f_3) - 10(f_3e_2 - f_2e_3) \\ -Q_2 &= -10.654 + 10(e_2e_3 + f_2f_3) - 4(f_3e_2 - f_2e_3). \end{aligned}$$

We should include

$$(1.1249)^2 = e_2^2 + f_2^2.$$

For bus 3 we have

$$-2 + j1 = (e_3 - jf_3)[(-4 + j5) + (-4 + j10)(e_2 + jf_2) + (8 - j15)(e_3 + jf_3)].$$

Separating real and imaginary parts, we get

$$\begin{aligned} -2 &= -4e_3 + 5f_3 - e_3(4e_3 + 10f_2) + f(10e_2 - 4f_2) + 8\left(e_3^2 + f_3^2\right) \\ 1 &= 5e_3 + 4f_3 + e_3(10e_2 - 4f_2) + f_3(4e_2 - 10f_2) - 15\left(e_3^2 + f_3^2\right). \end{aligned}$$

3.5.2.3.2 Case B: Formulation of the N–R Method in Polar Form

The PF problems using the N–R method can be formulated in polar coordinates. For any bus p, we have

$$V_p = |V_p|e^{j\delta p}, \quad \text{then } V_p^* = |V_p|e^{-j\delta p} \tag{3.66}$$

$$V_q = |V_q|e^{j\delta p}, \quad \text{then } Y_{pq} = |Y_{pq}|e^{-j\theta pq}, \tag{3.67}$$

where

δ is the phase angle of the bus voltages

θ_{pq} is the admittance angle

As a result, for any bus p,

$$P_p - jQ_p = V_p^* \sum_{q=1}^{n} Y_{pq} V_q. \tag{3.68}$$

Substituting Equation 3.105 into Equation 3.106, we have

$$P_p - jQ_p = \sum_{q=1}^{n} |V_p V_q Y_{pq}| e^{-j(\theta_{pq} + \delta_p - \delta_q)}. \tag{3.69}$$

Thus

$$\begin{aligned} P_p &= \text{Real}\left[V_p^* \sum_{q=1}^{n} Y_{pq} V_q \right] \\ &= \sum_{q=1}^{n} |V_p V_q Y_{pq}| \cos(\theta_{pq} + \delta_p - \delta_q) \\ &= |V_p V_p Y_{pp}| \cos(\theta_{pp}) + \sum_{\substack{q=1 \\ q \neq p}}^{n} |V_p V_q V_{pq}| \cos(\theta_{pq} + \delta_p - \delta_q) \end{aligned} \tag{3.70}$$

and

$$\begin{aligned} Q_p &= \text{Imaginary}\left[V_p^* \sum_{q=1}^{n} Y_{pq} V_q \right] \\ &= \sum_{q=1}^{n} |V_p V_q Y_{pq}| \sin(Q_{pq} + \delta_p - \delta_q) \\ &= |V_p V_p Y_{pp}| \sin \theta_{pp} + \sum_{\substack{q=1 \\ q \neq p}}^{n} |V_p V_q Y_{pq}| \sin(\theta_{pq} + \delta_p - \delta_q), \end{aligned} \tag{3.71}$$

for $p = 1, \ldots, n-1$ as the nth bus is a slack bus.

Now the linear N–R Equation in polar form becomes

$$\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = \begin{bmatrix} J_1 & \vdots & J_2 \\ \cdots & \cdots & \cdots \\ J_3 & \vdots & J_4 \end{bmatrix} \begin{bmatrix} \Delta \delta \\ \Delta |V| \end{bmatrix}, \tag{3.72}$$

where J_1, J_2, J_3, and J_4 are the elements of Jacobian **J** which can be calculated from the power Equations 3.70 and 3.71 as follows:

Elements of J_1
Off-diagonal elements

$$\left.\frac{\partial P_p}{\partial \delta_p}\right|_{p\neq q} = |V_p V_q Y_{pq}| \sin(\theta_{pq} + \delta_p - \delta_q). \tag{3.73}$$

Diagonal elements

$$\frac{\partial P_p}{\partial \delta_p} = -\sum_{\substack{q=1 \\ q\neq p}}^{n} |V_p V_q Y_{pq}| \sin(\theta_{pq} + \delta_p - \delta_q). \tag{3.74}$$

Elements of J_2
Off-diagonal elements

$$\left.\frac{\partial P_p}{\partial |V_q|}\right|_{p\neq q} = |V_p Y_{pq}| \cos(\theta_{pq} + \delta_p - \delta_q). \tag{3.75}$$

Diagonal elements

$$\left.\frac{\partial P_p}{\partial |V_p|}\right| = 2|V_p Y_{pp}| \cos\theta_{pp} + \sum_{\substack{q=1 \\ q\neq p}}^{n} |V_q Y_{pq}| \cos(\theta_{pq} + \delta_p - \delta_q). \tag{3.76}$$

Elements of J_3
Off-diagonal elements

$$\frac{\partial Q_p}{\partial \delta_q} = -|V_p V_q Y_{pq}| \cos(\theta_{pq} + \delta_p - \delta_q). \tag{3.77}$$

Diagonal elements

$$\frac{\partial Q_p}{\partial \delta_p} = -\sum_{\substack{q=1 \\ q\neq p}}^{n} |V_p V_q P_{pq}| \cos(\theta_{pq} + \delta_p - \delta_q). \tag{3.78}$$

Elements of J_4
Off-diagonal elements

$$\left.\frac{\partial Q_p}{\partial |V_q|}\right|_{q\neq p} = |V_p Y_{pq}| \sin(\theta_{pq} + \delta_p - \delta_q). \tag{3.79}$$

Diagonal elements

$$\frac{\partial Q_p}{\partial |V_p|} = 2|V_p Y_{pp}| \sin\theta_{pp} + \sum_{\substack{q=1 \\ q\neq p}}^{n} |V_q Y_{pq}| \sin(\theta_{pq} + \delta_p - \delta_q). \tag{3.80}$$

The elements of the Jacobian are calculated based on the latest voltage estimate and calculated power. However, the algorithm is the same as that applied with the rectangular coordinates. Notably, the formulation in the polar coordinates takes less computational effort and also requires less memory space.

The real power P is less sensitive to changes in the voltage magnitude $\Delta|V|$ and similarly the reactive power Q is less sensitive to the changes in the phase angle δ. We can thus approximate Equation 3.72 as follows:

$$\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = \begin{bmatrix} J_1 & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & J_4 \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta|V| \end{bmatrix}. \tag{3.81}$$

In the case of a generator bus P and $|V|$ are given. The real power P for any bus p is given by

$$P_p = \text{Real}\left[V_p^* \sum_{q=1}^{n} Y_{pq} V_q \right]. \tag{3.82}$$

and also for bus p, we have

$$|V_p|^2 = e_p^2 + f_p^2, \tag{3.83}$$

where

V_p is the voltage magnitude

e_p and f_p are its real and imaginary components, respectively

The matrix equations relating the changes in bus powers and square of the bus voltage magnitudes to the changes in the real and imaginary components of the voltages are

$$\begin{bmatrix} \Delta P \\ \cdots \\ \Delta Q \\ \cdots \\ \Delta|V|^2 \end{bmatrix} \begin{bmatrix} J_1 & \vdots & J_2 \\ \cdots & \cdots & \cdots \\ J_3 & \vdots & J_4 \\ \cdots & \cdots & \cdots \\ J_5 & \vdots & J_6 \end{bmatrix} \begin{bmatrix} \Delta e \\ \cdots \\ \Delta f \end{bmatrix}, \tag{3.84}$$

where

$$\Delta|V_p^k|^2 = \left[|V_p \text{ (scheduled)}|^2 - |V_p^k|^2 \right]$$

(V_p^k is the calculated bus voltage after the kth iteration.)

The elements of Jacobian are calculated as follows:

Elements of J_5
Off-diagonal elements

$$\frac{\partial |V_p|^2}{\partial e_q} = 0, \quad \text{for } q \neq p. \tag{3.85}$$

Diagonal elements

$$\frac{\partial |V_p|^2}{\partial e_p} = 2e_p. \tag{3.86}$$

Elements of J_6
Off-diagonal elements

$$\frac{\partial |V_p|^2}{\partial f_q} = 0, \quad \text{for } q \neq p. \tag{3.87}$$

Diagonal elements

$$\frac{\partial |V_p|^2}{\partial f_p} = 2f_p, \tag{3.88}$$

where

V_p^k is the bus voltage calculated at the kth iteration

V_p (scheduled) is the voltage given (i.e., specified) at any bus p as it is the generation bus

Calculations for the elements J_1, J_2, J_3, and J_4 were discussed earlier. After obtaining bus voltages, PF and line losses are calculated. We next give a modified Newton method, which has several benefits over the conventional N–R method.

Example 3.5.4

For the network shown in Figure 3.8, do the following:

1. Compute the *Y*-bus admittance matrix and state the initial bus voltages in polar form.
2. Write down the load-flow equations of the problem using the polar formulation.

Solution

1. We have in polar form:
   The diagonal terms of the Y_{bus}

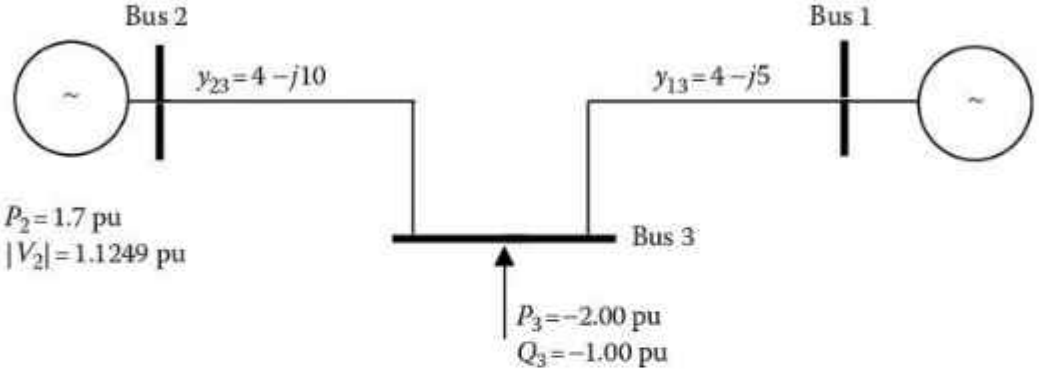


FIGURE 3.8
Single-line diagram.

$$Y_{11} = 6.4031\angle -51.34^\circ$$
$$Y_{22} = 10.77\angle -68.199^\circ$$
$$Y_{33} = 17.00\angle -61.928^\circ.$$

The off-diagonal terms of the Y_{bus}

$$Y_{12} = 0$$
$$Y_{13} = 6.4031\angle 128.66^\circ$$
$$Y_{23} = 10.77\angle 111.80^\circ.$$

Furthermore, $Y_{ij} = Y_{ji}$ since the admittance matrix is inherently symmetrical.
The bus voltages are represented as

$$V_1 = 1\angle 0$$
$$V_2 = 1.1249\angle \theta_2$$
$$V_3 = |V_3|\angle \theta_3.$$

1. We now can write the PFE for the three-bus system shown in Figure 3.8.
For bus 1, we have

$$P_1 - jQ_1 = V_1^*(Y_{11}V_1 + Y_{12}V_2 + Y_{13}V_3)$$
$$\Rightarrow P_1 - jQ_1 = 6.4031\angle -51.34^\circ + 6.4031|V_3|\angle 128.66 + \theta_3.$$

For bus 2, we have

$$P_2 - jQ_2 = V_2^*(Y_{12}V_1 + Y_{22}V_2 + Y_{23}V_3)$$
$$\Rightarrow 1.7 - jQ_2 = 13.628\angle -68.199^\circ + 12.115V_3\angle 118 + \theta_3 - \theta_2.$$

For bus 3, we have

$$P_3 - jQ_3 = V_3^*(Y_{13}V_1 + Y_{23}V_2 + Y_{33}V_3)$$
$$\Rightarrow -2 + j1 = V_3\angle -\theta_3[6.4031\angle 128.66^\circ + 10.77\angle 111.8^*1.1249$$
$$\angle \theta_2 + 17\angle -61.928^* V_3\angle \theta_3].$$

This reduces to

$$-2 + j1 = 6.4031V_3\angle 128.66^\circ + 12.11V_3\angle 111.8 - \theta_2 - \theta_2 + 17.00V_3\angle -61.928^\circ.$$

We now separate the real and imaginary parts of the above equations such that:

For bus 1

$$P_1 = 4 + 6.4031V_3\cos(128.66 + \theta_3)$$
$$-Q_1 = -5 + 6.4031V_3\sin(128.66 + \theta_3).$$

For bus 2

$$1.7 = 5.6612 + 12.115V_3\cos(111.8 + \theta_3 - \theta_2)$$
$$-Q_2 = -12.653 + 12.115V_3\sin(111.8 + \theta_3 - \theta_2).$$

For bus 3

$$-2 = 6.4031V_3\cos(128.66 - \theta_3) + 12.115V_3\cos(111.8 + \theta_2 - \theta_3) + 8V_3^2$$
$$1 = 6.4031V_3\sin(128.6 - \theta_3) + 12.115V_3\sin(111.8 + \theta_2 - \theta_3) - 15V_3^2.$$

The above six equations define the PF problems in polar form.

3.5.3 Fast-Decoupled PF Method

The fast-decoupled PF method is a very fast technique for computing the PF solution, in which both speed and sparsity are exploited. The technique is an extension of Newton's method formulated in polar coordinates with certain approximations.

The earlier PFE (i.e., Equation 3.72) using the N–R method can be expressed in polar coordinates as

$$\begin{bmatrix}\Delta P\\ \Delta Q\end{bmatrix} = \begin{bmatrix}H & N\\ M & L\end{bmatrix}\begin{bmatrix}\Delta\delta\\ \frac{\Delta|V|}{V}\end{bmatrix}, \tag{3.89}$$

where H, N, M, and L are the elements (viz., J_1, J_2, J_3, and J_4) of the Jacobian matrix. Since changes in real power (i.e., ΔP) are less sensitive to the changes in voltage magnitude (ΔV) and changes in reactive power (i.e., ΔQ) are less sensitive to the changes in angle ($\Delta\delta$), Equation 3.89 is written as

$$\begin{bmatrix}\Delta P\\ \Delta Q\end{bmatrix} = \begin{bmatrix}H & O\\ O & L\end{bmatrix}\begin{bmatrix}\Delta\delta\\ \frac{\Delta|V|}{V}\end{bmatrix}. \tag{3.90}$$

Equation 3.90 consists of two decoupled equations:

$$[\Delta P] = [H][\Delta\delta] \tag{3.91}$$

and

$$[\Delta Q] = [L]\frac{[\Delta V]}{V}. \tag{3.92}$$

The elements of the Jacobian (Equation 3.90) are defined again (Equation 3.88),

$$H_{pq} = \frac{\delta P_p}{\delta\delta_q} \quad \text{and} \quad L_{pq} = \frac{\delta Q_p}{\delta E_q}|E_q|.$$

Equations 3.91 and 3.92 for calculating the elements of the Jacobian (H and L) are

$$\begin{aligned} P_p &= \sum_{q=1}^{n} |V_p V_q Y_{pq}| \cos[\theta_{pq} + \delta_p - \delta_q] \\ &= |V_p V_p Y_{pp}| \cos\theta_{pp} + \sum_{q=1}^{n} |V_p V_q Y_{pq}| \cos[\theta_{pq} + \delta_p - \delta_q] \end{aligned} \tag{3.93}$$

and

$$Q_p = |V_p V_p Y_{pp}| \sin\theta_{pp} + \sum_{\substack{q=1 \\ q\neq p}}^{n} |V_p V_q Y_{pq}| \sin[\theta_{pq} + \delta_p - \delta_q]. \tag{3.94}$$

The off-diagonal elements of H are

$$\begin{aligned} H_{pq} = \frac{\delta P_p}{\delta\delta_q} &= |V_p V_q Y_{pq}| \sin[\theta_{pq} + \delta_p - \delta_q] \\ &= |V_p V_q Y_{pq}| \left[\sin[\theta_{pq}]\cos[\delta_p - \delta_q] + \cos\theta_{pq}\sin[\delta_p - \delta_q]\right] \\ &= |V_p V_q| \left[Y_{pq}\sin\theta_{pq}\cos[\delta_p - \delta_q] + Y_{pq}\cos\theta_{pq}\sin[\delta_p - \delta_q]\right] \\ &= |V_p V_q| \left[-B_{pq}\cos[\delta_p - \delta_q] + G_{pq}\sin[\delta_p - \delta_q]\right]. \end{aligned} \tag{3.95}$$

Similarly, the off-diagonal elements of L are

$$\begin{aligned} L_{pq} = \frac{\delta Q_p |V_q|}{\delta V_q} &= |V_p V_q Y_{qp}| \sin[\theta_{pq} + \delta_p - \delta_q] \\ &= |V_p V_q| \left[G_{pq}\sin[\delta_p - \delta_q] - B_{pq}\cos[\delta_p - \delta_q]\right]. \end{aligned} \tag{3.96}$$

From Equations 3.95 and 3.96, it is evident that

$$H_{pq} = L_{pq} = |V_p V_q|\left[G_{pq}\sin[\delta_p - \delta_q] - B_{pq}\cos[\delta_p - \delta_q]\right]. \tag{3.97}$$

The diagonal elements of H are given by

$$\begin{aligned} H_{pp} = \frac{\delta P_p}{\delta \delta_p} &= -\sum_{\substack{q=1 \\ q\neq p}}^{n} |V_p V_q Y_{pq}| \sin[\theta_{pq} + \delta_p - \delta_q] \\ &= -\left[\sum_{q=1}^{n} |V_p V_q Y_{pq}| \sin[\theta_{pq} + \delta_p - \delta_q] - |\boldsymbol{V_p V_q Y_{pp}}| \sin\theta_{pp}\right] \\ &= [+Q_p + V_p^2 B_{pp}] \\ &= -V_p^2 B_{pp} - Q_p \end{aligned} \tag{3.98}$$

and the elements of L are given by

$$\begin{aligned} L_{pp} &= \frac{\delta Q_p |V_p|}{\delta V_p} \\ &= |2V_p^2 Y_{pp}| \sin\theta_{pp} + \sum_{\substack{q=1 \\ q\neq p}}^{n} |V_p V_q Y_{pq}| \sin[\theta_{pq} + \delta_p - \delta_q] \\ &= |2V_p^2 Y_{pp}| \sin\theta_{pp} + Q_p - |V_p^2 Y_{pp}| \sin\theta_{pp} \\ &= Q_p + = |V_p^2 Y_{pp}| \sin\theta_{pp} \\ &= -V_p^2 B_{pp} + Q. \end{aligned} \tag{3.99}$$

In the case of fast-decoupled PF formulation, the following approximations are made.

$$\cos[\delta_p - \delta_q] \approx 1$$

$$G_{pq}\sin[\delta_p - \delta_q] \ll B_{pq}$$

and

$$Q_p \ll B_{pp} V_p^2. \tag{3.100}$$

These approximations are made for the calculations of elements of Jacobian H and L, to yield

$$H_{pq} = L_{pq} = -|V_p||V_q|B_{pq} \quad \text{for } q \neq p$$

and

$$H_{pp} = L_{pp} = -B_{pq}\left|V_p^2\right|. \tag{3.101}$$

Equations 3.91 and 3.92 with the substitutions of elements of Jacobian Equation 3.99 take the form

$$\Delta P = H\Delta\delta,$$

that is,

$$[\Delta P_p] = [V_p][V_q][B'_{pq}][\Delta\delta_q] \tag{3.102}$$

and similarly

$$[\Delta Q_p] = [V_p][V_q][B''_{pq}][\Delta V], \tag{3.103}$$

where B'_{pq} and B''_{pq} are the elements of the [-B] matrix.

Further decoupling and the final algorithm for fast-decoupled PF studies are obtained by the following:

1. Omit from B' the representation of those network elements that affect MVAr flows, that is, shunt reactances and off-nominal in-phase transformer taps.
2. Omit from B'' the angle-shifting affects of phase-shifters.
3. Divide Equations 3.102 and 3.103 by V_p set $V_q = 1$ pu, and also neglect the series resistances in calculating the elements of B'.

With these assumptions, Equations 3.102 and 3.103 take the following final form:

$$\left[\frac{\Delta P_p}{V_p}\right] = [B'][\Delta\delta] \tag{3.104}$$

and

$$\left[\frac{\Delta Q_p}{V_p}\right] = [B'][\Delta V], \tag{3.105}$$

where both [B'] and [B''] are real and sparse and have the same structure as H and L, respectively. Since they contain network admittances, they are constant and need to be triangularized only once at the beginning of the iterations. This algorithm results in a very fast solution of $\Delta\delta$ and ΔV.

3.5.4 Linearized (DC) PF Method

The resistance of transmission lines is rather small compared to the reactances of those lines. Then the resistances are neglected to simplify the

solutions. Considering the nominal π model of a medium transmission line, we can represent the line with its inductive reactance only. Then the transmission line parameters (*ABCD*) will be

$$\begin{aligned} A &= D = 1\angle 0 \\ B &= jX_{ik},\ C = 0 \end{aligned} \tag{3.106}$$

and the PFs through the line are

$$\begin{aligned} P_S &= -\frac{V_S V_R}{X}\cos(90^\circ + \delta) = \frac{V_S V_R}{X}\sin(\delta) \\ P_R &= -\frac{V_S V_R}{X}\cos(90^\circ - \delta) = \frac{V_S V_R}{X}\sin(\delta) \end{aligned} \tag{3.107}$$

for small angles, $\sin(\delta) = \delta$. Hence, the line flows become

$$\begin{aligned} P_{ik} &= \frac{V_i V_k}{X_{ik}}\delta_{ik}, \\ \delta_{ik} &= \delta_i - \delta_k. \end{aligned} \tag{3.108}$$

Since all bus voltages of a power system are around 1 pu, let

$$\begin{aligned} V_i &= V_k = 1\,\text{pu}, \\ b_{ik} &= \frac{-1}{X_{ik}}. \end{aligned} \tag{3.109}$$

Then Equation 3.108 becomes

$$P_{ik} = -b_{ik}\delta_{ik} = \frac{\delta_i - \delta_k}{X_{ik}}. \tag{3.110}$$

Now, the bus power at any bus is the sum of the PFs in the lines connected to that bus. Hence

$$P_i = \sum_{k=1}^{N} P_{ik} = \sum_{k=1}^{N} -b_{ik}\delta_{ik} \quad i = 1, 2, \ldots, N \tag{3.111}$$

or in matrix form

$$\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{bmatrix} = -\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1N} \\ b_{21} & b_{22} & \cdots & b_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ b_{N1} & b_{N2} & \cdots & b_{NN} \end{bmatrix}\begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_N \end{bmatrix}. \tag{3.112}$$

which can be abbreviated as

$$[P] = [b][\delta],$$
$$b_{ik} = \frac{-1}{X_{ik}}, \quad b_{ii} = \sum_{k=1}^{N} (-b_{ik}). \tag{3.113}$$

That is, the matrix

$$[\delta] = [b]^{-1}[[P]]. \tag{3.114}$$

Until now, we have kept the system ground as the reference bus. However, since we have adopted all shunt branches in simplifying things, we have lost our reference. This means that the matrix $[b]$ of Equation 3.112 obtained by Equation 3.113 will be a singular matrix. Hence $[b]^{-1}$ does not exist. To overcome this difficulty, we select one of the buses as reference and assign zero radian to its angle (as in the swing bus for ac PF). Then, all the calculated angles will be referred to this bus, and the row and column corresponding to this bus in $[b]$ will be dropped to produce a nonsingular $[b]$ matrix.

Note that since the system is linearized, the solution is direct and there is no need for an iterative procedure.

3.6 Practical Applications of PF Studies

The way in which PF studies are used encompasses two system areas: system planning and system operation. System planning has the objective of designing a system capable of providing reliable bulk electric supply. The common tasks of the system planner include transmission planning (the design, analysis, and sizing of future transmission circuits), interchange studies, generation adequacy studies, and cost-to-benefit analyses of system additions. Generation planning is also considered as part of system planning. System operation aims at setting, adjusting, and operating the system to produce a reliable and economical electric energy supply. Included in system operation tasks are economic dispatch of the generating stations (i.e., calculating the power levels at each generating unit so that the system is operated most economically), contingency analysis (analysis of outages and other forced operating conditions), and studies that ensure power pool coordination.

In most phases of each of these areas, load-flow studies are used to assess system performance under given operating conditions. An important application of PF studies involves transmission planning. Models of the future system are prepared and, typically, peak load conditions are used to run a study. The intention is to find the proper size components (conductors, transformers, reactors, and shunt capacitors), and, in addition, transmission facility siting may be considered. In some cases, series capacitors, phase-shifters,

and tap-changing under loads (TCULs) are studied where control of PF is important. A second general area of system planning in which PF studies are indispensable is interchange studies. Interties with neighbors are planned to meet predicted needs of the system consistent with reliability requirements. Load-flow studies identify line loads and bus voltages out of range, inappropriately large bus phase angles (and the potential for stability problems), component loads (principally transformers), proximity to Q-limits at generation buses, and other parameters that have the potential of creating operating difficulties. While peak load studies are frequently performed, sometimes intermediate loads and off-peak (minimum) load conditions are used. Off-peak loads may result in high-voltage conditions that are not identified during the peak load.

3.6.1 Case Study Discussion

System operation tasks that utilize PF studies retaining all bus voltage and transmission component loads within range involve loss calculations, area coordination, calculation of fixed-tap settings, and various types of contingency checks. The latter checks the effects of line and component outages (sometimes multiple contingencies are considered). Contingency studies are usually done offline (i.e., in advance of operating the system under the conditions considered). A typical line outage contingency study consists of a base-case PF study (with all lines in service) followed by contingency cases in which key lines are outaged. Key lines are identified in a variety of ways: lines that are heavily loaded or have a small load margin (rating minus operating load), lines with high-angular difference in terminal bus voltage phases, extra high-voltage circuits, or lines that are known to the operators to be essential to system integrity. The latter come from system operating experience and are often the most reliable means of identifying key lines. In some cases, approximate PF studies may be sufficient to identify problems in contingency studies—several suitable approximate methods are presented in Chapter 4. Contingency studies are particularly important to maintain reliable service. If a line outage causes other circuits to become heavily loaded, those circuits may trip out by the action of protective relays. These additional outages can result in further outages in an uncontrolled way. The term cascade tripping applies to this undesirable operating condition.

3.7 Illustrative Examples

Example 3.7.1

Find all the solutions to the following two nonlinear simultaneous equations

$$4x_1 + 3x_2^2 = 0$$
$$6x_1x_2 + 2x_2 = 0.$$

Solution

To find all the solutions, we can use the first equation to define x_1 as a function of x_2, $x_1 = ((-3)/4)x_2^2$. By substituting in the second equation, we get

$$6\left(\frac{-3}{4}x_2^2\right)x_2 + 2x_2 = 0.0 \Rightarrow 2x_2\left(1 - 9x_2^2\right) = 0.0,$$

which can be factored as follows

$$2x_2[(1 - 3x_2)(1 + 3x_2)] = 0.0,$$

which has three possible solutions, namely

$$x_2 = 0.0, \quad x_2 = \frac{1}{3}, \quad \text{and} \quad x_2 = \frac{-1}{3}.$$

The available solutions are

1. $x_2 = 0.0$, $x_1 = 0.0$
2. $x_2 = \frac{1}{3}$, $x_1 = \frac{-3}{4}\left(\frac{1}{9}\right) = \frac{-1}{12}$
3. $x_2 = \frac{-1}{3}$, $x_1 = \frac{-3}{4}\left(\frac{1}{9}\right) = \frac{-1}{12}$

Then the solution points are

$$(0.0)', \quad \left(\frac{-1}{12}, \frac{1}{3}\right)', \quad \text{and} \quad \left(\frac{-1}{12}, \frac{-1}{3}\right)'.$$

Example 3.7.2

The circuit elements in the 138 kV circuit in Figure 3.9 are in per unit on a 100 MVA base with the nominal 138 kV voltage as base. The $(P + jQ)$ load is scheduled to be 170 MW and 50 MVAr.

1. Write the Y matrix for the two-bus system.
2. Assume that bus 1 is a reference bus and set up the G–S correction equation for bus 2.
3. Use a flat start on bus 2. Carry out two or three iterations and show that you are converging.

Solution

1. The line impedance of this line is $z_{12} = 0.01 + j0.04$ (pu). Then the line admittance can be expressed as

$$y_{12} = 1/z_{12} = \frac{1}{(0.01 + j0.04)} = 5.88 - j23.53 \text{ (pu)}.$$

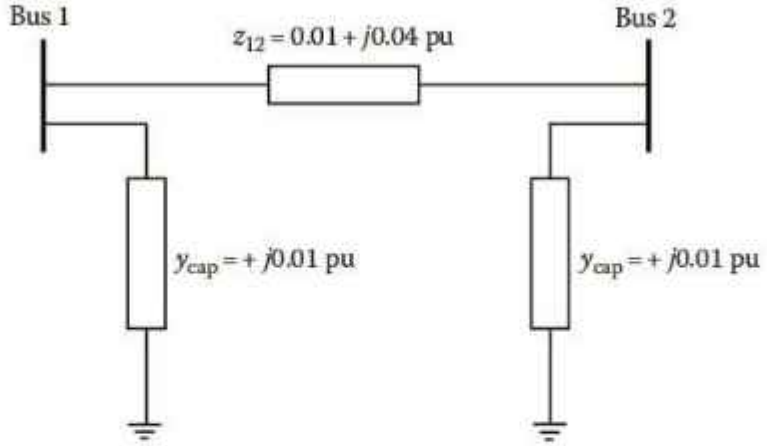


FIGURE 3.9
One-line diagram for Example 3.7.2.

The bus admittance matrix components can be expressed as follows.

$$Y_{11} = y_{12} + y_{10} = (5.88 - j23.53) + (j0.01) = 5.88 - j23.54$$
$$Y_{22} = y_{12} + y_{20} = (5.88 - j23.53) + (j0.01) = 5.88 - j23.54$$
$$Y_{12} = y_{21} + -y_{12} = -(5.88 - j23.53).$$

Then the Y-matrix can be expressed as

$$Y_{bus} = \begin{bmatrix} 5.88 - j23.54 & -5.88 + j23.53 \\ -5.88 + j23.53 & 5.88 - j23.54 \end{bmatrix}.$$

2. Taking bus 1 as reference, $V_1 = 1.0\angle 0.0$ (pu).
The G–S equation for bus 2 is

$$V_2^{(n+1)} = \frac{1}{Y_{22}}\left[\frac{P_2 - jQ_2}{V_2^{*(n)}} - Y_{21}V_1^{(n)}\right].$$

We have the load $P_2 - JQ_2 = (1.7 - j0.5)$ (pu).

First iteration
Taking as an initial value $V_2^{(0)} = 1.0\angle 0.0$ (pu),

$$V_2^{(1)} = \frac{1}{Y_{22}}\left[\frac{P_2 - jQ_2}{V_2^{*(0)}} - Y_{21}V_1\right]$$
$$V_2^{(1)} = \frac{1}{(5.88 - j23.54)}\left[\frac{(1.7 - j0.5)}{1.0\angle 0.0} - (-5.88 + j23.53)(1.0\angle 0.0)\right]$$
$$= 1.039\angle 3.48^\circ \text{ (pu)}.$$

Second iteration

Now

$$V_2^{(1)} = 1.039\angle 3.48^\circ \text{ (pu)}$$

$$V_2^{(2)} = \frac{1}{Y_{22}}\left[\frac{P_2 - jQ_2}{V_2^{*(1)}} - Y_{21}V_1\right]$$

$$V_2^{(2)} = \frac{1}{(5.88 - j23.54)}\left[\frac{(1.7 - j0.5)}{(1.039)\angle -3.48^\circ} - (-5.88 + j23.53)(1.0\angle 0.0)\right]$$

$$= 1.04\angle 3.21^\circ \text{ (pu)}.$$

Third iteration

Now

$$V_2^{(2)} = 1.04\angle 3.21^\circ \text{ (pu)}.$$

$$V_2^{(3)} = \frac{1}{Y_{22}}\left[\frac{P_2 - jQ_2}{V_2^{*(2)}} - Y_{21}V_1\right]$$

$$V_2^{(2)} = \frac{1}{(5.88 - j23.54)}\left[\frac{(1.7 - j0.5)}{1.04\angle -3.21^\circ} - (-5.88 + j23.53)(1.0\angle 0.0)\right]$$

$$= 1.0402\angle 3.22^\circ \text{ (pu)}.$$

As a result, we can say $V_2^{(1)} \approx V_2^{(2)}$.

Then $V_2 = 1.0402\angle 3.22^\circ$ (pu).

Example 3.7.3

Consider the simple electric power system shown in Figure 3.10 and carry out the following calculations. Write down the elements of the bus admittance matrix Y.

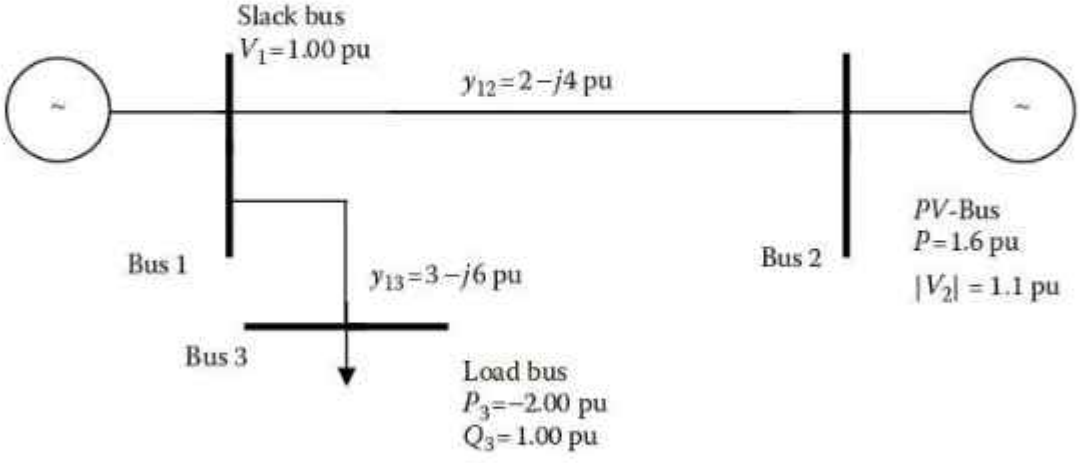


FIGURE 3.10
System for illustrative Example 3.7.3 (Section 3.7).

1. Using the active power equation at bus 2. Calculate the phase angle θ_2.
2. Using the active and reactive power equations at bus 3. Calculate $|V_3|$ and hence θ_3
3. Calculate the active real power at bus 1.
4. Find the total active power loss in the system.

Solution

1. Calculation of the terms of the Y_{BUS} yields

$$\begin{aligned} Y_{11} &= 5 - j10, & Y_{22} &= 2 - j4 \\ &= 11.18\angle -63.43, & &= 4.47\angle -63.43 \\ Y_{33} &= 3 - j6, & Y_{12} &= -2 + j4 \\ &= 6.71\angle -63.43, & &= 4.47\angle -116.57 \end{aligned}$$

$$\begin{aligned} Y_{13} &= -3 + j6 \\ &= 6.71\angle 116.57. \end{aligned}$$

At bus 2

$$P_2 = |V_2|[Y_{12}V_1\cos(\theta_2 - \theta_1 - \Psi_{12}) + Y_{22}V_2\cos(\theta_2 - \theta_2 - \Psi_{22}) + Y_{23}V_3\cos(\theta_2 - \theta_3 - \Psi_{23})]$$

and thus

$$1.6 = 1.1[(4.47)(1.0)\cos(\theta_2 - 116.57) + (4.47)(1.1)\cos(-63.43)].$$

As a result

$$\begin{aligned} &\cos(\theta_2 - 116.57) = -0.1669 \\ &\theta_2 - 116.57 = \pm 99.59535 \\ &\theta_2 = (216.16^\circ) \text{ or } (16.97465^\circ). \end{aligned}$$

Take $\theta_2 = 16.97465^\circ$.
For bus 3, we have

$$P_2 = |V_3|[|Y_{31}||V_1|\cos(\theta_3 - \theta_1 - \Psi_{31}) + |Y_{33}||V_3|\cos(-\Psi_{33})].$$

By substituting, we get

$$-2 = |V_3|[6.71*1*\cos(\theta_3 - 116.57) + 6.71V_3\cos(63.43)].$$

Thus

$$\frac{-2}{6.71} = V_3^2\cos 63.43 + V_3\cos(\theta_3 - 116.57).$$

Also we have

$$Q_3 = |V_3|[|Y_{31}||V_1|\sin(\theta_3 - \theta_1 - \Psi_{31}) + |Y_{33}||V_3|\sin(-\Psi_{33})]$$
$$1 = |V_3|[6.71\sin(\theta_3 - 116.57) + 6.71V_3\sin(63.43)]$$
$$\frac{1}{6.71} = |V_3|^2\sin 63.43 + |V_3|\sin(\theta_3 - 116.57).$$

Combining Equations (a) and (b),

$$\left[\frac{2}{6.71} + |V_3|^2\cos(63.43)\right]^2 + \left[\frac{1}{6.71} + |V_3|^2\sin(63.43)\right]^2 = |V_3|^2$$

or

$$|V_3|^4 + \frac{4}{6.71}|V_3|^2[\cos(63.43) + 0.5\sin(63.43)] + \frac{5}{(6.71)^2} = |V_3|^2;$$

this gives

$$|V_3|^4 - |V_3|^2 + \frac{1}{a} = 0$$
$$|V_3|^2 = \frac{1 \pm \sqrt{5/3}}{2}.$$

Take the positive sign: $|V_3|^2 = 0.8727$.

The solution for $|V_3|$ is $|V_3| = 0.9342$.

As a result $-2/6.71 = 0.8727\cos(63.43) + 0.9342\cos(\theta_3 - 116.57)$. Thus $\cos(\theta_3 - 116.57) = -0.7369014307$, or $\theta_3 - 116.57 = \pm 137.468$. As a result $\theta_3 = -20.898$. We now obtain P_1 as

$$P_1 = |V_1|[|Y_{11}||V_1|\cos(-\Psi_{11}) + |Y_{12}||V_2|\cos(\theta_1 - \theta_2 - \Psi_{12}) + |Y_{13}||V_3|\cos(\theta_1 - \theta_3 - \Psi_{13})]$$
$$P_1 = 0.9937.$$

3.8 Conclusion

The chapter discussed the PF problem as an important tool for the operation of the existing power system as well as planning for expected expansion. It started by classifying the different buses in an electric power system network and defined the quantities associated with each of the specific types in Section 3.2. Section 3.3 presented the general forms of the PFEs in an electric power system. Section 3.4 gave insights on potential modeling considerations in the generation subsystem, transmission subsystems, and load modeling. The formulation was extended to handle transformer and phase-shifter inclusion in the PF model; the formulation in both polar and rectangular

coordinates was also presented in this section. The different iterative techniques used to solve the PF problem including the G–S iterative technique and the N–R method were presented in Section 3.5. Special forms of load-flow solution methods such as fast-decoupled PF were presented at the end of this section. Some practical applications of the power-flow studies and some illustrative examples to the power-flow problem were presented in Section 3.5. The formulation in both polar and rectangular coordinates was also presented followed by some unsolved exercises.

3.9 Problem Set

PROBLEM 3.9.1

Given the following equations

$$f_1(x) = x_1 - x_2^2 + x_2 = 0$$
$$f_2(x) = 2x_2^2 - x_1x_2 - 4 = 0.$$

Obtain their solutions using the Jacobian method?

PROBLEM 3.9.2

Consider the following nonlinear simultaneous equations

$$f_1(x) = x_1^2 + x_2 = 0$$
$$f_2(x) = x_1 + x_2 = 0.$$

1. Solve the equations graphically.
2. Solve f_1 for x_2 and f_2 for x_1, solve the equations by the Jacobian method with $x_1 = 0.5$ as an initial guess.
3. Show how Part 2 converges graphically.

PROBLEM 3.9.3

A one-line diagram of a three-bus power system is given in Figure 3.11, where the reactances are in per units on a 100 MVA base. Use G–S iteration to obtain the PF solution within a tolerance of 0.01 pu,

$$V_1 = 1.0\angle 0 \text{ pu}$$
$$V_2 = 1.0\angle 0 \text{ pu}$$
$$P_2 = 0.6 \text{ pu}$$
$$P_3 = -1.0 \text{ pu}$$
$$Q_3 = -0.75 \text{ pu (lagging)}.$$

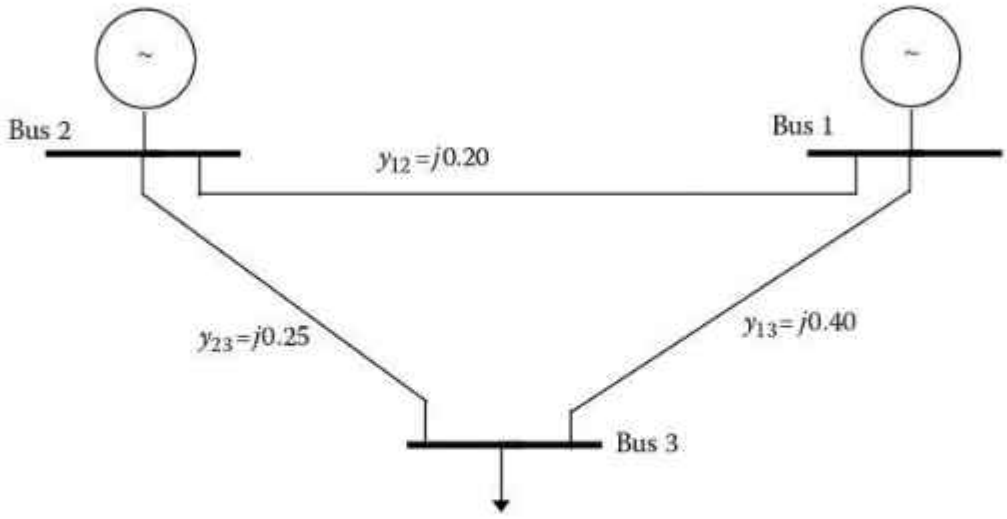


FIGURE 3.11
One-line diagram for Problem 3.9.3.

PROBLEM 3.9.4

Do one N–R iteration of the power system of Problem 3.9.3?

PROBLEM 3.9.5

Using the solution to Problem 3.3, calculate the PF through lines 1–3 at both ends?

PROBLEM 3.9.6

Consider a six-bus power system as shown in Figure 3.12. The line impedances are in per units on a 100 MVA, 138 kV system base and the generator at the slack or reference bus operates at rated voltage. Table 3.6 shows the system load for the six-bus power system.

1. Solve the PF problem for the given system using the G–S approach. Calculate the total system losses in the transmission network.
2. Consider the loss of line L5–6. Resolve the PF problem and the transmission losses.
3. Compare the results of Part (1) and Part (2). Explain the effect of the line outage on the voltage profile of the system.

PROBLEM 3.9.7

Given the network shown in Figure 3.13 (base 100 MVA), do the following:

1. Calculate the phase angles for the set of power injections?

   $P_1 = 100$ MW generation
   $P_2 = 120$ MW load
   $P_3 = 150$ MW generation
   $P_4 = 200$ MW load.

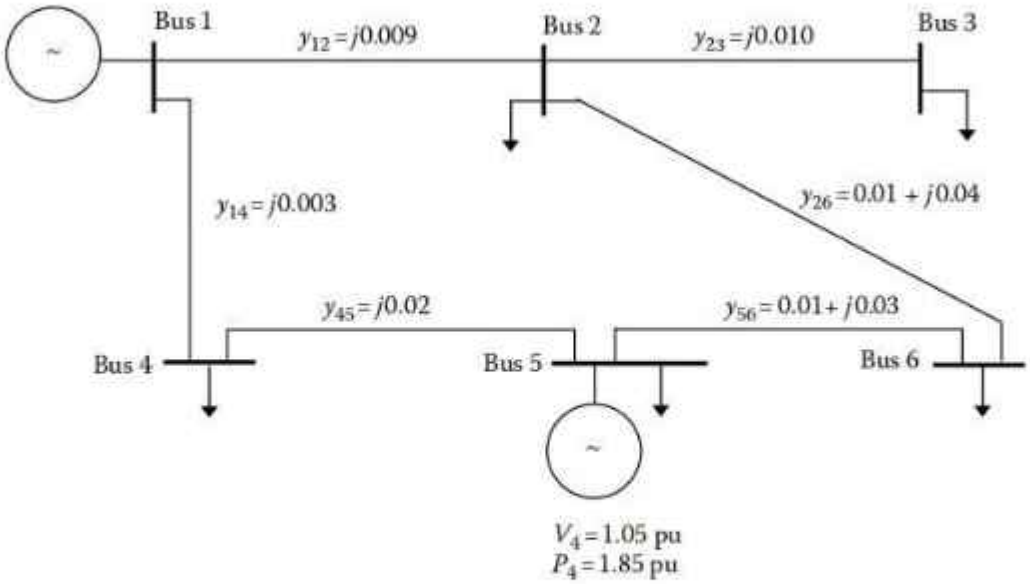


FIGURE 3.12
Simplified six-bus power system.

TABLE 3.6

Load Profile for the Six-Bus Power System

	Load	
Bus ID	**Real Power, P_i (MW)**	**Reactive Power, Q_i (MVAr)**
1	0	0
2	85	60
3	15	8.0
4	60	37
5	10	−4.0
6	36	24

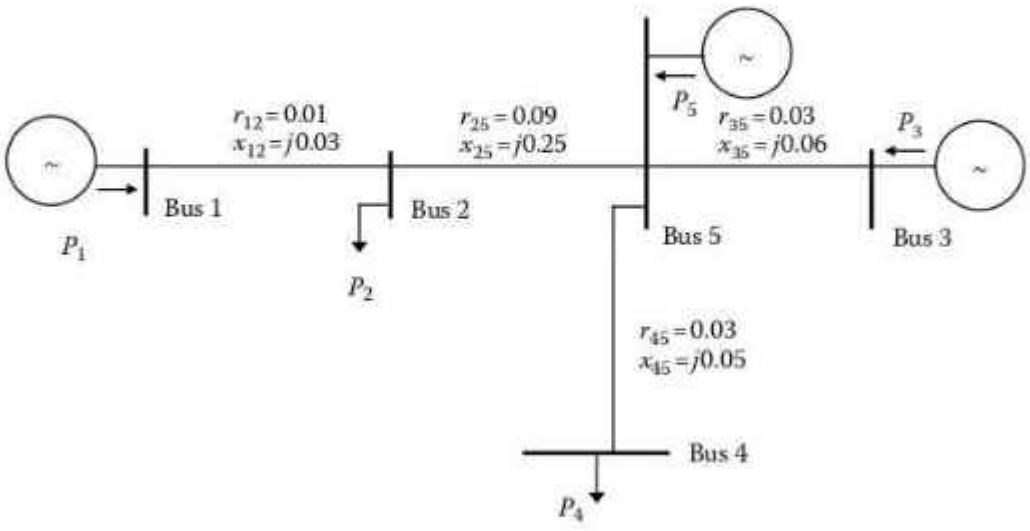


FIGURE 3.13
Four-bus network for Problem 3.9.7.

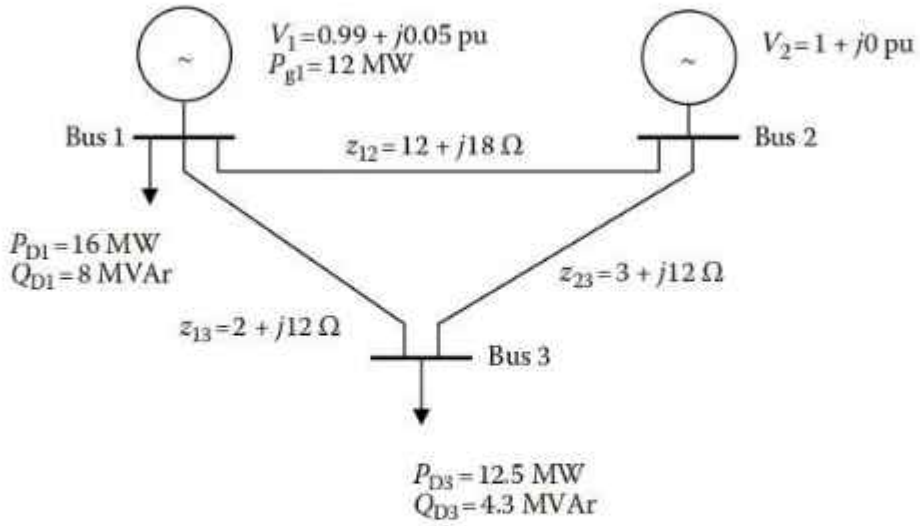


FIGURE 3.14
Three-bus network for Problem 3.9.8.

2. Calculate P_5 according to the decoupled load flow?
3. Calculate all PFs on the system using the phase angles found in Part I?
4. Optional: Calculate the reference bus penalty factors for buses through 4. Assume all bus voltage magnitudes are 1.0 pu?

PROBLEM 3.9.8

For the network shown in Figure 3.14 solve the PF problem using the N–R method.

This part of the network is working on 138 kV.

References

1. Elgerd, O. I., *Electric Energy Systems Theory—An Introduction*, McGraw-Hill, New York, 1982.
2. El-Hawary, M. E., *Electric Systems, Dynamics and Stability with Artificial Intelligence Applications*, Marcel Dekker, New York, 1999.
3. Grainger, J. J. and Stevenson, W. D., Jr., *Power System Analysis*, McGraw-Hill, New York, 1994.
4. Gross, C. A., *Power Systems Analysis*, Wiley, New York, 1979.
5. Pai, M. A., *Computer Techniques in Power System Analysis*, Tata McGraw-Hill, New Delhi, India, 1979.
6. Weedy, B. M., *Electric Power Systems*, 3rd edn., Wiley, London, 1979.

4

Constrained Optimization and Applications

4.1 Introduction

Practical system problems are very often formulated with some constraints imposed on their variables. Optimization of such problems must be carried out within the limits of these constraints. The basic difference between the constrained and the unconstrained problems is the selection of admissible points that are eligible for optimization. The so-called admissible points are those vectors that satisfy the constraints of the problem. An optimum of the constrained problem is judged by the comparison of admissible points. Constrained problems are assumed to have at least one admissible point.

Lagrange multipliers are known as an effective means of dealing with constraints. Two theorems for functions are given. The first theorem in each case gives a necessary condition and the second theorem gives a sufficient condition for minimization of constrained problems. Again, we apply the sufficiency theorems to test for maximum of $f(y)$ by minimizing $f(y)$. Procedures for solving the constrained problems by using the theorems are listed and also utilized in the solutions of the illustrative problems [1,2].

Before we continue further, I wish to introduce some necessary background in analytic computation that will be useful in optimization work. First, it includes matrix notation and some special properties that have significant applications in operation research and power systems in particular. Basic foundations in matrix algebra and functional calculus are required by the reader and the author has introduced standard background operations to test negative and positive semidefinite conditions on Jacobian and Hessian matrices.

Now, consider the n-dimensional before given by $\mathbf{y}(t) = [y(t)_1, y(t)_2, \ldots, y(t)_n]^{\mathrm{T}}$ where each element $y_i(t)$ is continuous in time and y is special scalar when $n = 1$. In this chapter, we limit our attention to case when y is independent of t. Then, $n \times m$ matrix $\mathbf{M}$ is now defined as $\mathbf{M} = [\mathbf{y}_j]$ where the vector construct of the matrix is $\mathbf{M} = [\mathbf{y}_j \mid j = 1,m] = [y_{i,j}]_{n \times m}$.

By definition, if A is square and Hermitian such that its $\mathbf{M} = (\mathbf{M}^{\mathrm{T}})^* = \mathbf{M}^{\dagger}$, then $\mathbf{M}$ is positive definite if, for all nonzero n-dimensional vectors $\mathbf{x}$, the inner product is strictly positive. That is to say $\langle \mathbf{Mx}, \mathbf{x} \rangle > 0$. And, $\mathbf{M}$ is positive semidefinite if, under the same conditions, $\langle \mathbf{Mx}, \mathbf{x} \rangle \geq 0$. (Reversing the signs concludes a parallel definition for negative definite and negative semidefinite conditions, respectively.)

Alternatively, **M** is said to be positive semidefinite if all its eigenvalues are positive and $\langle \mathbf{Mx},\mathbf{x} \rangle \equiv \mathbf{x}^T\mathbf{Mx} > 0$. This definition is particularly important in characterizing the convex behaviors of many nonlinear systems, and in particular, those with quadratic models.

Other computational considerations of matrices in vector notation used in the text include the formulations of the following:

1. $n \times n$ Jacobian matrix or $J(\mathbf{f}(\mathbf{x}))$

   The Jacobian matrix is the array of all first-order partial derivatives of a vector-valued function such that:

$$J(x) = \nabla(f(x)) = \left[\frac{\partial f_i}{\partial x_j}\right]_{n\times n}, \quad \text{where } i \in \{1,n\} \text{ and } i \in \{1,n\}$$

2. Development of the Hessian $H(\mathbf{x})$ matrix

   Consider the matrix product given by

$$\mathbf{y}^T J(\mathbf{f}(\mathbf{x}))\mathbf{y} = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial^2 f}{\partial x_i x_j} y_i y_j\right)$$

   (quadratic form of the matrix product)

   The form is positive definite if $\mathbf{y}^T J(x)\mathbf{y} > 0$ for all real values of z_i and z_j. The test for this condition can be found by evaluating a set of determinants of $J(\mathbf{y}(\mathbf{x}))$. The Hessian is now defined at the determinant of the Jacobian matrix J and is the nth principal determinant of J given by

$$H(\mathbf{f}(\mathbf{x})) = |J(\mathbf{f}(\mathbf{x}))| = \left|\left[\frac{\partial^2 f}{\partial x_i x_j}\right]\right|_{n\times n}, \quad \text{where } i \in \{1,n\} \text{ and } i \in \{1,n\}$$

   (The corresponding principal subdeterminants are called sub-Hessians.)

   The symmetric matrix **A** is said to be negative definite if its corresponding form is negative definite $\mathbf{A} = \mathbf{y}^T J(x)\mathbf{y} < 0$. In general, the positive (or negative) definite tests on the symmetric Hessian matrix are used in the criteria for local minima (or maxima) to be discussed later in subsequent sections of the chapter.

4.2 Theorems on the Optimization of Constrained Functions [5,6]

The constrained problem is defined to optimize the scalar function $f(y)$ subject to constraints

$$f_i(y) = K_i, \quad \text{for } i = 1, 2, \ldots, m, \tag{4.1}$$

where all $f_i(y)$ are real, scalar, and single-valued functions of y. Although the problem is formulated with equality constraints, problems with inequality constraints can be solved by varying the parameters K_i after the optimization.

The Lagrange function for the problem is defined to be the scalar function

$$L(y) = f(y) + \lambda F(y), \tag{4.2}$$

where

$$\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_m] \tag{4.3}$$

and

$$F(y) = [f_1(y), \ldots, f_m(y)]^T. \tag{4.4}$$

The components λ_i of the row vector λ are constant and are known as the Lagrange multipliers. Note that the Lagrange function is a function of y and λ.

4.2.1 Continuity Assumption

All the functions involved in the following theorems are assumed to be continuous at x which is an optimum under consideration. For simplicity, the arguments of the partial derivatives of the Lagrange and constraint functions evaluated at x are dropped. That is, $L_x = L_x(x)$, $L_{xx} = L_{xx}(x)$, and $F_x = F_x(x)$ are used in the statement without proof of the theorems.

4.2.2 Theorems

THEOREM 4.2.1

If x is an optimum of the constrained problem, then it is necessary that $L_x = 0$ for some λ.

$$L_x = f_x + \lambda F_x = 0. \tag{4.5a}$$

The Lagrange multipliers designate the coefficients of linear combination.

THEOREM 4.2.2

If x is a candidate solution as determined by the necessary conditions of Theorem 4.2.1, then the sufficiency conditions that must be satisfied are

$$T = L_{xx}(x, \lambda) + \beta F_x^{\mathrm{T}}(x) F_x(x) > 0 \tag{4.5b}$$

for some positive semidefinite values of β (i.e., $\beta \geq 0$).

The significance of the theorem lies in the facts that (i) the constrained problem can be extremized without considering the constraints and (ii) the nonnegative definite matrix $F_x^{\mathrm{T}} F_x$ that is added to L_{xx} in the second theorem enables one to test for sufficiency without considering the constraints.

4.3 Procedure for Optimizing Constrained Problems (Functions)

There are four steps involved in solving constrained problems by applying the theorems:

Step 1: *Formulation of the Lagrange function*
The inequality constraint $K_{i1} \leq f_i(y) \leq K_{i2}$ is assumed to be $f_i(y) = K_i$ with $K_{i1} \leq K_i \leq K_{i2}$ considered as constant during the optimization. Form $F(y) = [f_1(y), f_2(y), \ldots, f_m(y)]^{\mathrm{T}}$ and $L = f(y) + \lambda F(y)$, where $f(y)$ is to be optimized.

Step 2: *Determination of optimum candidate(s)*
Set $L_x = 0$ (Theorem 4.2.1), which means replace y by x and then differentiate with respect to x. Here it is understood that L_x is continuous at x. There are altogether $n + m$ equations; n from $L_x = 0$ and m from the constraints that can be obtained to solve for the optimum candidate x and Lagrange multipliers λ_i.

Step 3: *Sufficiency test*
Assume that all the functions involved in the following are continuous at x. Then the candidate x obtained in step 2 is a minimum (Theorem 4.2.2) if the square matrix

$$L_{xx} + \beta F_x^{\mathrm{T}} F_x$$

is positive definite for $\beta \geq 0$. It is a maximum if the matrix is positive definite after replacing L_{xx} by $-L_{xx}$ (actually L by $-L$).

Step 4: *Further optimization*
The function $f(x)$ is now a function of K_i. Further optimization is to be sought with respect to K_i in the interval $K_{i1} \leq K_i \leq K_{i2}$ for all i. The optimum of K_i may occur at the boundaries of the constraints.

4.4 Karush–Kuhn–Tucker Condition

Also known as Khun–Tucker or KKT conditions, they are developed to providing unique tests of the necessary and sufficiency conditions of optimality. It is well developed for nonlinear programming that involves the use of the Lagrangian approach. A special treatment of the concepts to be described shortly is also applicable to linear programming problems by taking advantage of duality theorems [6–10].

Recall the generalized NLP

Min **f(x)**

subject to

$g_i(x) = 0 \quad \text{and} \quad h_j(x) \geq 0,$

where $f(\mathbf{x})$ is the scalar function of the n-vector $\mathbf{x}$ to be minimized. And, for the set of equality constraints, $i \in \{1, p\}$; for the set of inequality constraints, $j \in \{1,m\}$. We now summarize the following conditions of KKT.

KKT conditions: If their should exist a point x^* at a local minimum of $f(x)$, there exist constants λ_i and μ_j where $i \in \{1,p\}$ and $j \in \{1,m\}$, respectively, such that

1. $\nabla f(x^*) - \sum_{i=1}^{p} \lambda_i \nabla g_i(x^*) - \sum_{j=1}^{m} \mu_j \nabla h_j(x^*) = 0$
2. $g_i(x^*) = 0$ for all $i \in \{1,p\}$
3. $h_j(x^*) \geq 0$ for all $j \in \{1,m\}$
4. $\mu_j \geq 0$ for all $j \in \{1,m\}$
5. $\mu_j h_j(x^*) = 0$ for all $j \in \{1,m\}$

This KKT conditions present a new Lagrange multiplier μ_j that corresponds to the dual variables in linear programming with economic implications based on the problem being solved. It should be noted that condition 1 is the necessary condition for x^* to be a candidate, conditions 2 and 3 are the constraints qualifications, and conditions 4 and 5 are eliminated nonoptimal solutions to guarantee feasibility, if they exist.

4.5 Illustrative Problems

PROBLEM 4.5.1

Given a function $f(y) = y_1 - y_2^2$, find an optimum subject to constraint $y_1 y_2 \geq 2$.

Solution

1. $f_1(y) = y_1 y_2 = K$ with $2 \leq K \leq \infty$ and $L = y_1 - y_2^2 + \lambda y_1 y_2$.
2. $L_x = [1 + \lambda x_2, \quad -2x_2 + \lambda x_1] = 0$ gives $1 + \lambda x_2 = 0$ and $-2x_2 + \lambda x_1 = 0$.
   Solution of these yields $x_1 = -2\lambda^{-2}$, $x_2 = -\lambda^{-1}$, and $K_1 = 2\lambda^{-3}$.
3. $$L_{xx} + \beta F_x^{\mathrm{T}} F_x = \begin{bmatrix} 0 & \lambda \\ \lambda & -2 \end{bmatrix} + \beta \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} [x_2 \quad x_1]$$
   $$= \begin{bmatrix} \beta x_2^2 & \beta x_1 x_2 + \lambda \\ \beta x_1 x_2 + \lambda & \beta x_1^2 - 2 \end{bmatrix}.$$
   The matrix has a negative determinant for any $\beta \geq 0$ and, hence, it cannot be positive definite. Let us try for maximum by replacing L_{xx} by $-L_{xx}$. That is
   $$L_{xx} + \beta F_x^{\mathrm{T}} F_x = \begin{bmatrix} \beta x_2^2 & \beta x_1 x_2 - \lambda \\ \beta x_1 x_2 - \lambda & \beta x_1^2 + 2 \end{bmatrix}$$
   $$= \begin{bmatrix} \beta \lambda^{-2} & 2\beta \lambda^{-3} - \lambda \\ 2\beta \lambda^{-3} - \lambda & 4\beta \lambda^{-4} + 2 \end{bmatrix}.$$
   It is positive definite for $\beta > \lambda^4/6$ and, therefore, the solution of step 2 is a maximum.
4. Maximum function is
   $$f(x) = -2\lambda^{-2} - \lambda^{-2} = -3\lambda^{-2} = -3\left(\frac{1}{2}K\right)^{2/3},$$
   where $2 \leq K \leq \infty$. Further maximization yields $K = 2$ for which $f(x) = -3$, $x_1 = -2$, and $x_2 = -1$.

PROBLEM 4.5.2

Optimize $f(y) = y^{\mathrm{T}} P y + C^{\mathrm{T}} y$ subject to linear constraints $Ay = d$ where P is a symmetric n-square and nonsingular matrix. The matrix A is of $m \times n$ with full rank and C and d are n- and m-vectors.

Solution

1. Constraints are $Ay = d$ and, hence
   $$L = y^{\mathrm{T}} P y + C^{\mathrm{T}} y + \lambda A y.$$

2. $L_x = 2x^T P + C^T + \lambda A = 0$. The solution of x and λ is

$$x = -\frac{1}{2} P^{-1}(A^T \lambda^T + C),$$

where $\lambda^T = -(AP^{-1}A^T)^{-1}\,(AP^{-1}C + 2d)$.

3. $L_{xx} + \beta F_x^T F_x = 2P + \beta A^T A$. The solution of step 2 is a minimum if $2P + \beta A^T A > 0$ and is a maximum if $2P + \beta A^T A > 0$ for some $\beta \geq 0$.

4. $f(x) = \frac{1}{4}(\lambda A - C^T)P^{-1}(A^T \lambda^T + C)$.

   (a) Minimum norm *of* $Ay = d$ with $m \leq m$. The problem is a special case of $P = 1$ and $C = 0$. It follows from the result of step 2 that

$$\lambda^T = -2(AA^T)^{-1} d \quad \text{and} \quad x = A^T(AA^T)^{-1} d.$$

   The sufficiency for the minimum is assured by choosing $\beta = 0$. The minimized norm is $f(x) = x^T x = d^T\,(AA^T)^{-1} d$.

   (b) Least square approximation for $Ax = d$ with mn. The problem is to minimize, without constraint,

$$\begin{aligned} f(y) = e^T e &= (Ax - d)^T (Ax - d) \\ &= x^T A^T A x - 2d^T A x + d^T d. \end{aligned}$$

   Let $P = A^T A$ and $C = -2A^T d$; then the result of step 2 yields

$$x = (A^T A)^{-1} A^T d \quad \text{and} \quad e^T e = d^T \left[I - A(A^T A)^{-1} A^T \right] d.$$

   The norm is a minimum because A is of rank n and hence $A^T A > 0$. The sufficiency is assured again by choosing $\beta = 0$.

4.5.1 Nonpower Systems Application Examples

PROBLEM 4.5.3

Consider the function

$$f(x_1, x_2, x_3) = x_1 + 2x_3 + x_2 x_3 - x_1^2 - x_2^2 - x_3^2.$$

Solution

Applying the necessary condition $\nabla f(X_0) = 0$, this gives

$$\frac{\partial f}{\partial x_1} = 1 - 2x_1 = 0.$$

$$\frac{\partial f}{\partial x_2} = x_3 - 2x_2 = 0.$$

$$\frac{\partial f}{\partial x_3} = 2 + x_2 - 2x_3 = 0.$$

The solution of these simultaneous equations is given by

$$X_0 = \left(\frac{1}{2}, \frac{2}{3}, \frac{4}{3}\right).$$

Another way to check the sufficiency condition is to check the Hessian matrix **H** for positive or negative definiteness. Thus

$$H|x_0 = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_3} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \dfrac{\partial^2 f}{\partial x_2 \partial x_3} \\ \dfrac{\partial^2 f}{\partial x_3 \partial x_1} & \dfrac{\partial^2 f}{\partial x_3 \partial x_2} & \dfrac{\partial^2 f}{\partial x_3^2} \end{bmatrix}_{X_0}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

The principal minor determinants $H|_{X_0}$ have the values -2, 4, and -6, respectively. Thus, $H|_{X_0}$ is negative definite and

$$X_0 = \left(\frac{1}{2}, \frac{2}{3}, \frac{4}{3}\right)$$

represents a maximum point.

4.6 Power Systems Application Examples

4.6.1 Optimal Operation of an All-Thermal System: Equal Incremental Cost-Loading

A simple, yet extremely useful problem in optimum economic operation of electric power systems is treated here. Consider the operation of m thermal generating units on the same bus as shown in Figure 4.1 [4]. Assume that the variation of the fuel cost of each generator (F_i) with the active-power output (P_i) is given by a quadratic polynomial. The total fuel cost of the plant is the sum of the individual unit cost converted to $\$/h$:

$$F = \sum_{i=1}^{m} \alpha_i + \beta_i P_i + \gamma_i P_i^2, \tag{4.6}$$

where α_i, β_i, and γ_i are assumed available.

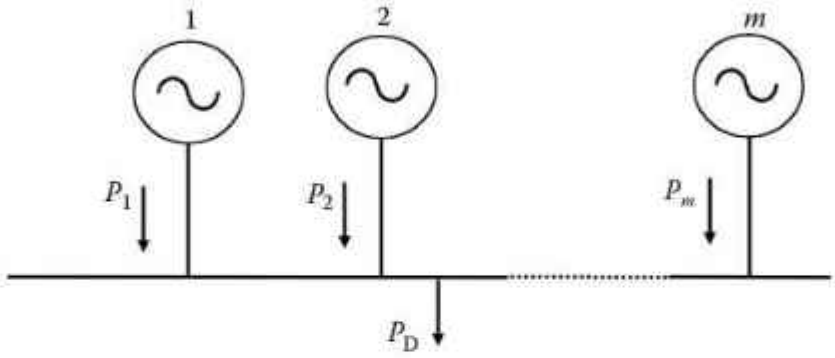


FIGURE 4.1
Units on the same bus.

We wish to determine generation levels such that F is minimized while simultaneously satisfying an active-power balance equation. This utilizes the principle of power-flow continuity. Here, the network is viewed as a medium of active-power transfer from the generating nodes to the load node. Only one equation is needed.

The first active-power balance equation model neglects transmission losses and, hence, we can write

$$P_D = \sum_{i=1}^{m} (P_i) \tag{4.7}$$

with P_D being a given active-power demand for the system.

The demand P_D is the sum of all demands at load nodes in the system. The model is useful in the treatment of parallel generating units at the same plant since in this case the negligible transmission losses assumption is valid.

We write the constraint equation (Equation 4.7)

$$P_D - \sum_{i=1}^{m} (P_i) = 0. \tag{4.8}$$

The technique is based on including Equation 4.8 in the original cost function by use of a Lagrange multiplier, say λ, which is unknown at the outset. Thus

$$L = F_T + \lambda \left[P_D - \sum_{i=1}^{m} (P_i) \right], \tag{4.9}$$

where

$$F_T = \sum_{i=1}^{m} [F_i(P_i)].$$

Note that λ is to be obtained such that Equation 4.8 is satisfied. The idea here is to penalize any violation of the constraint by adding a term corresponding to the resulting error. The Lagrange multiplier is, in effect, a conversion factor that accounts for the dimensional incompatibilities of the cost function ($/h) and constraints (MW). The resulting problem is unconstrained, and we have increased the number of unknowns by one.

The optimality conditions are obtained by setting the partial derivatives of L with respect to P_i to 0. Thus

$$\frac{\partial F_i}{\partial P_i} - \lambda = 0. \tag{4.10}$$

Note that each unit's cost is independent of the generations of other units.

The expression obtained in Equation 4.10 leads to the conclusion that

$$\lambda = \frac{\partial F_1}{\partial P_1} = \frac{\partial F_2}{\partial P_2} = \cdots. \tag{4.11}$$

The implication of this result is that for optimality, individual units should share the load such that their incremental costs are equal. We can see that the λ is simply the optimal value of incremental costs at the operating point. Equation 4.10 is frequently referred to as the equal incremental cost-loading principle.

Implementing the optimal solution is straightforward for the quadratic cost case where we have

$$F_i(P_i) = \alpha_i + \beta_i P_i + \gamma_i P_i^2.$$

Our optimality conditions from Equation 4.10 reduce to

$$\beta_i + 2\gamma_i P_i - \lambda = 0. \tag{4.12}$$

The value of λ is determined such that Equation 4.8 is satisfied. This turns out to give

$$\lambda = \frac{2P_D + \sum_{i=1}^{m} (\beta_i/\gamma_i)}{\sum_{i=1}^{m} \gamma_i^{-1}}. \tag{4.13}$$

Finally, using Equation 4.12 the optimal generations are obtained as

$$P_i = \frac{\lambda - \beta_i}{2\gamma_i}. \tag{4.14}$$

4.6.2 Optimal Operation of an All-Thermal System, Including Losses

We are interested in minimizing the total cost given by Equation 4.6 while satisfying the active-power balance equation including losses. Thus

$$P_D = \sum_{i=1}^{m} (P_i) - P_L, \tag{4.15}$$

where P_L is the active-power loss considered as a function of the active-power generation alone as outlined in the previous section. Following our treatment for the loss-free case, we form the augmented cost function:

$$\hat{F} = F_T + \lambda \left[P_D + P_L - \sum_{i=1}^{m} (P_i) \right]. \tag{4.16}$$

The optimality conditions are obtained using the same arguments as before and are

$$\frac{\partial F_i}{\partial P_i} + \lambda \left(\frac{\partial P_L}{\partial P_i} - 1 \right) = 0. \tag{4.17}$$

Note that with negligible transmission losses, the above expression reduces to Equation 4.19.

It is convenient to transform the obtained optimality expression into an equivalent form. This is done by defining the factors L_i:

$$L_i = \left(1 - \frac{\partial P_L}{\partial P_i} \right)^{-1}. \tag{4.18}$$

We can write Equation 4.17 as

$$L_i \frac{\partial F_i}{\partial P_i} = \lambda \ (i = 1, \ldots, m). \tag{4.19}$$

This is of the form of Equation 4.11 except for the introduction of the new factors L_i, which account for the modifications necessitated by including the transmission loss. These are traditionally called the penalty factors to indicate that plant costs (F_i) are penalized by the corresponding incremental transmission losses ($\partial P_L / \partial P_i$).

Examination of Equation 4.19 reveals that the optimal generations are obtained when each plant is operated such that the penalized incremental costs are equal.

Flowcharts for the two cases of economic dispatch are shown in Figure 4.2.

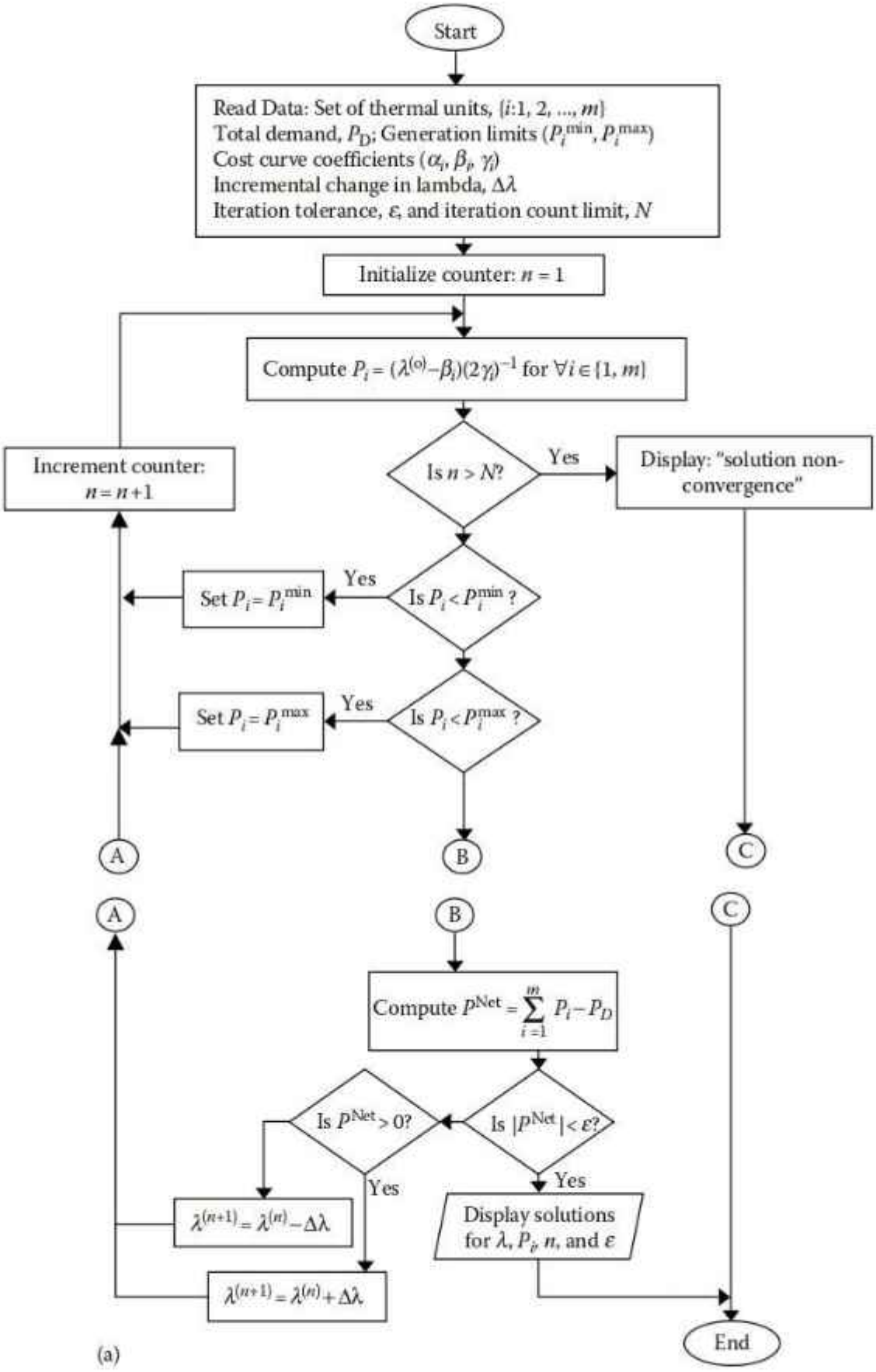
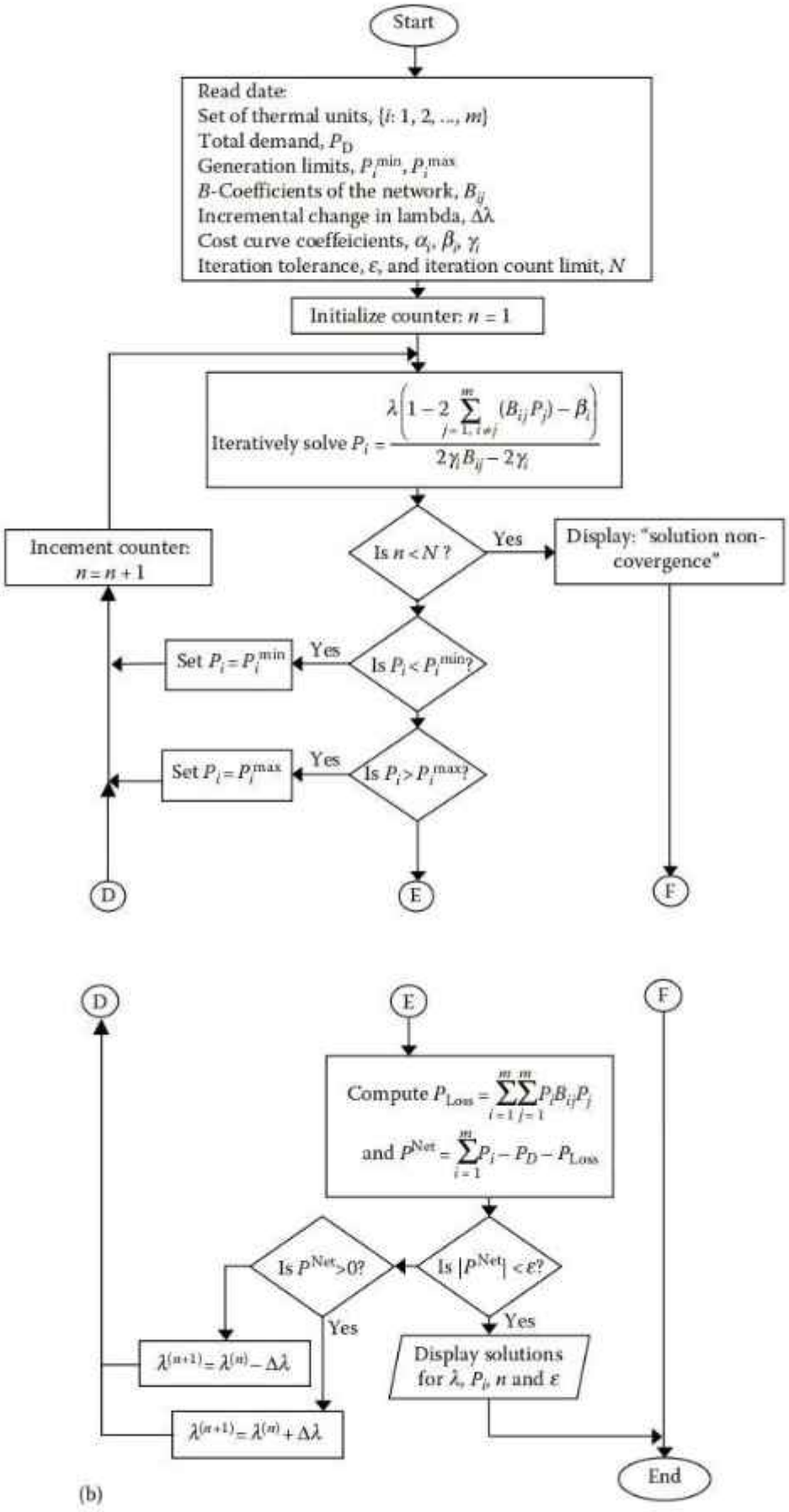


FIGURE 4.2
Economic dispatch flow chart. (a) Neglecting the contributions of transmission losses, and



(b)

FIGURE 4.2 (continued)
(b) accounting for the effects of transmission losses.

PROBLEM 4.6.1

A certain power system consists of two generating plants and a load. The transmission losses in the system can be approximated by the following quadratic equation.

$$P_{\text{Loss}} = 0.606 \times 10^{-3} P_1^2 + 0.496 \times 10^{-3} P_2^2.$$

The cost functions of the units are modeled such that

$$\beta_1 = 10.63 \quad \gamma_1 = 3.46 \times 10^{-3}$$
$$\beta_2 = 12.07 \quad \gamma_2 = 3.78 \times 10^{-3}.$$

Given that the output of plant 2 is 445 MW while it is being operated under optimal conditions, determine the

1. Incremental cost of operating the units, assuming the equal cost sharing
2. Output of plant 1
3. Total power losses
4. Efficiency of the system

Solution

1. For economic dispatch (optimal power flow, see Figure 4.2) at plant 2,

$$\frac{\partial F_i(P_i)}{\partial P_i} = \lambda \left(1 - \frac{\partial P_{\text{Loss}}}{\partial P_i}\right)\Bigg|_{i=2},$$

where

$$F_2(P_2) = \alpha_2 + \beta_2 P_2 + \gamma_2 P_2^2,$$
$$\beta_2 + 2\lambda_2 P_2 = \lambda \left(1 - \frac{\partial P_{\text{Loss}}}{\partial P_2}\right)$$
$$\Rightarrow \lambda = \frac{\beta_2 + 2\lambda_2 P_2}{(1 - (\partial P_{\text{Loss}}/\partial P_2))}$$
$$= \frac{12.07 + 2(3.78 \times 10^{-3})(445)}{1 - 2(0.496 \times 10^{-3})(445)}$$
$$\lambda = 27.63 \text{ (\$/MW h)}.$$

2. Similarly, for the optimal operation of plant 1:

$$\beta_1 + 2\lambda_1 P_1 = \lambda \left(1 - \frac{\partial P_{\text{Loss}}}{\partial P_1}\right)$$

$$\begin{aligned}\beta_1 + 2\lambda_1 P_1 &= \lambda(1 - 2\beta_{11} P_1)\\ \Rightarrow P_1 &= \frac{\lambda - \beta_1}{2(\gamma_1 + \lambda\beta_{11})}\\ &= \frac{27.63 - 10.63}{2(0.00346 + (27.63)(0.606 \times 10^{-3}))}\\ &= 421 \text{ MW}.\end{aligned}$$

3. Hence, the total power losses due to transmission are

$$\begin{aligned}P_{\text{LOSS}} &= 0.606 \times 10^{-3} \times (421)^2 + 0.496 \times 10^{-3} \times (445)^2\\ &= 107 + 98.2\\ &= 107 + 98.2\\ &= 206 \text{ MW}.\end{aligned}$$

4. Efficiency of the system is given as

$$\begin{aligned}\eta &= \frac{P_{\text{output}}}{P_{\text{input}}}\\ &= 1 - \left(\frac{P_{\text{LOSS}}}{\sum_i P_i}\right)\\ &= 1 - \left(\frac{206 \text{ MW}}{(421 + 445) \text{ MW}}\right)\\ &= 0.763\% \text{ or } 76.3\%.\end{aligned}$$

4.7 Illustrative Examples

Example 4.7.1

Find the local and global minima of the function

$$f(x) = f(x_1, x_2) = x_1^2 + x_2^2 - 2 \times 1 - 4x_2 + 5|x_1|,\ |x_2| \leq 3.$$

Solution

$$\min F(x) = x_1^2 + x_2^2 - 2x_1 - 4x_2 + 5$$

$$|x_1| \leq 3 \rightarrow -3 \leq x_1 \leq 3$$

s.t.

$$|x_2| \leq 3 \rightarrow -3 \leq x_2 \leq 3.$$

We can change the constraint to be an equality constraint

$$x_1 - k_1, \quad -3 \le k_1 \le 3$$
$$x_2 = k_2, \quad -3 \le k_2 \le 3.$$

1. From the Lagrange function.

$$L = x_1^2 + x_2^2 - 2x_1 - 4x_2 + 5 + \lambda_1 x_1 + \lambda_2 x_2.$$

2. Determine optimum candidates:

$$\frac{\partial L}{\partial x_i} = 0$$
$$\frac{\partial L}{\partial x_1} = 2x_1 - 2 + \lambda_1 = 0 \rightarrow x_1 = 1 - 0.5\lambda_1$$
$$\frac{\partial L}{\partial x_2} = 2x_2 - 4 + \lambda_2 = 0 \rightarrow x_2 = 2 - 0.5\lambda_2.$$

3. Sufficiency test.

$$L_{xx} + \beta F_x^T F_x > 0 \text{ for some positive } \beta,$$
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \ 1] = \begin{bmatrix} 2+\beta & \beta \\ \beta & 2+\beta \end{bmatrix} = 4 + 4\beta > 0 \text{ for any } \beta.$$

Then it is at a minimum.

4. Further optimization.

$$K_1 = K_2 = K, \quad \text{as } -3 \le K \le 3$$
$$F(K) = K^2 - 2K - 4K + 5 = 2K^2 - 2K + 5.$$

For minimization $F_K = 4K - 2 \rightarrow K = 0.5$.

$$F_{KK} = 2 \text{ (i.e., it is minimum)}$$
$$K = 0.5 = X_1 = X_2, \quad f_{\min} = 4.5$$
$$f(K) = 4.5.$$

Example 4.7.2

Two thermal units at the same station have the following cost models.

$$F_1 = 793.22 + 7.74P_1 + 0.00107P_1^2$$
$$F_2 = 1194.6 + 7.72P_2 + 0.00072P_2^2$$
$$100 \le P_2 \le 800 \text{ MW} \quad \text{and} \quad 100 \le P_1 \le 600 \text{ MW}.$$

Find the optimal power generated P_1 and P_2 and the incremental cost of power delivered for demands of 400, 600, and 1000 MW, respectively.

SOLUTION

$$\frac{\partial F_1}{\partial P_1} = 7.74 + 0.00214P_1$$

$$\frac{\partial F_2}{\partial P_2} = 7.72 + 0.00144P_2.$$

For optimality

$$\frac{\partial F_1}{\partial P_1} = \frac{\partial F_2}{\partial P_2}$$

$$7.74 + 0.00214P_1 = \lambda \rightarrow P_1 = 467.29(\lambda) - 3616.8$$

$$7.72 + 0.00144P_2 = \lambda \rightarrow P_2 = 694.44(\lambda) - 5361.1$$

$$P_D = P_1 + P_2.$$

1. At $P_D = 400$ MW, $P_1 + P_2 = 400$.

$$400 = 467.29\lambda - 3626.8 + (694.44\lambda - 5361.1)$$

$$9377.91 = 1161.73\lambda \rightarrow \lambda = 8.072$$

$P_1 = 155.3$ MW, $P_2 = 244.7$ MW;
both of them are within the desired limit.

2. At $P_D = 600$ MW, $P_1 + P_2 = 600$ MW.

$$600 = 467.29(\lambda) - 3616.8 + (694.44(\lambda) - 5361.1)$$

then

$\lambda = 8.245$, $P_1 = 235.78$ MW, $P_2 = 364.22$ MW;
both of them are within the desired limit.

3. At $P_D = 1000$ MW, $P_1 + P_2 = 1000$ MW.

$$1000 = 467.29(\lambda) - 3616.8 + (6944(\lambda) - 5361.1)$$

then

$\lambda = 8.589$, $P_1 = 396.7$ MW, $P_2 = 603.33$ MW;
both of them are in the desired limit.

Example 4.7.3

The following fuel cost equations model a plant consisting of two thermal units.

$$F_1 = 0.00381P_1^2 + 9.2P_1 + 70.5 \text{ (\$)}$$
$$F_2 = 0.00455P_2^2 + 6.0P_2 + 82.7 \text{ (\$)}.$$

1. For a load of 600 MW, determine the optimal power generated by each unit and the equal incremental fuel cost λ at which it operates.
2. For the same load as in Part 1, given that the generation is constrained as

$$80.0 \leq P_1 \leq 250 \text{ MW}$$
$$120 \leq P_2 \leq 400 \text{ MW}$$

at what values of λ should the units be operated?

Solution

1. From the given data,

$$\lambda_1 = 0.00381, \quad \beta_1 = 9.2, \quad \alpha_1 = 70.5$$
$$\lambda_2 = 0.00455, \quad \beta_2 = 6.0, \quad \alpha_2 = 82.7.$$

By the formula for the incremental fuel cost,

$$\lambda = \frac{2P_D + \sum(\beta_i/\gamma_i)}{\sum(\gamma_i)^{-1}},$$

where

$$\sum \frac{\beta_i}{\lambda_i} = \frac{9.2}{0.00381} + \frac{6.0}{0.00455} = 3.733 \times 10^4$$
$$\sum \frac{1}{\lambda_i} = \frac{1}{0.00381} + \frac{1}{0.00455} = 4.822 \times 10^2$$
$$\therefore \lambda = \frac{2(600) + 3.733 \times 10^4}{4.822 \times 10^2}$$
$$= 10.23 \text{ \$ MW h},$$

at which point, $(\partial F_i/\partial P_i) = \lambda$, for all i, such that

$$2(0.00381)P_1 + 9.2 = 10.23$$
$$\Rightarrow P_1 = 135 \text{ MW}.$$

Similarly, $2(0.00455)P_2 + 6.0 = 10.23 \Rightarrow P_2 = 465$ MW.

Notably, $P_D = P_1 + P_2$ (the power balance equation).

2. For equal incremental fuel costing, there is a violation on the given constraint. Therefore, since P_2 is greater than P_{2max}, let P_2 assume its upper bound 200 MW. Thus

$$P_1 = P_D - P_2 = (600 - 400)$$
$$= 200 \text{ MW},$$

which is within the desired limits.

The incremental cost for each unit is calculated from $(\partial F_1/\partial P_i) = \lambda_i$, for all is

$$\lambda_1 = 0.00762P_1 + 9.2 = (0.00762 \times 200) + 9.2 = 10.72 \text{ \$/MW h}$$

and

$$\lambda_2 = 0.00910P_2 + 6.0 = (0.00910 \times 400) + 6.0 = 9.64 \text{ S/MW h}.$$

Example 4.7.4

Consider a power system that is modeled by two generating plants such that their cost function parameters are

$$\beta_1 = 5.93 \quad \gamma_1 = 4.306 \times 10^{-3}$$
$$\beta_2 = 6.02 \quad \gamma_2 = 4.812 \times 10^{-3}.$$

Given also that the network has the following B-coefficients,

$$B_{11} = 3.95 \times 10^{-4}$$
$$B_{22} = 4.63 \times 10^{-4},$$

the system load is 700 MW and the constraints on the generation are

$$100 \leq P_{g1} \leq 500 \text{ MW}$$
$$80 \leq P_{g1} \leq 500 \text{ MW}.$$

1. Calculate the incremental cost and the optimal value for the plants' output.
2. Neglecting the transmission losses, repeat Part 1 while considering the following generation constraints.

SOLUTION

1. Generally, the quadratic approximation of the cost-function modeling the ith plant is

$$F_i(P_i) = \alpha_i + \beta_i P_i + \gamma_i P_i^2,$$

and the transmission losses can be expressed as

$$P_{\text{Loss}} = \sum_{i=1}^{m} \sum_{j=1}^{m} (P_i B_{ij} P_j),$$

where
m is the number of units
B_{ij} is the loss coefficients

For economic dispatch of power from the units with equal cost sharing:

$$\frac{\partial F_i(P_i)}{\partial P_i} = \lambda \left(1 - \frac{\partial P_{\text{Loss}}}{\partial P_i}\right).$$

From the data given, we observe that

$$P_{\text{Loss}} = B_{11} P_1^2 + B_{22} P_2^2; \quad B_{12} = 0.$$

Therefore, for plant 1: $-\beta_1 + 2\gamma_1 P_1 = \gamma(1 - 2B_{11}P_1)$ and for plant 2: $\beta_2 + 2\gamma_2 P_2 = \lambda(1 - 2B_{22}P_2)$,

$$\Rightarrow \frac{\beta_1 + 2\gamma_1 P_1}{\beta_2 + 2\gamma_2 P_2} = \frac{1 - 2B_{11}P_1}{1 - 2B_{22}P_2}.$$

Using $P_2 = P_D - P_1$, further manipulation of the loss equation with substitution of the constants gives

$$P_1^2 + 7.5897 \times 10^4 P_1 - 2.86767 \times 10^7 = 0 \Rightarrow P_1 = 376 \text{ MW}$$

$$\therefore P_2 = P_D - P_1 = 700 - 376 = 324 \text{ MW}.$$

Now, from the equation for plant 1

$$\beta_1 + 2\gamma_1 P_1 = \lambda(1 - 2B_{11}P_1) \Rightarrow \lambda = \frac{\beta_1 + 2\gamma_1 P_1}{1 - 2B_{11}P_1}$$

$$= \frac{5.93 + (2 \times 4.306 \times 10^{-3} \times 3760)}{1 - (2 \times 3.95 \times 10^{-4} \times 376)} = \frac{9.16811}{0.70296} = 13.042 \text{ \$ MW h}.$$

The incremental cost for the plants is 13.042 \$ MW.

2. Neglecting the transmission losses implies that $(\partial P_{\text{Loss}}/\partial P_i) = 0$. Hence, for optimal power flow (economic dispatch) at each plant

$$\frac{\partial F_T}{\partial P_i} = 0$$

$$\beta_i + 2\gamma_i P_i = \lambda_i, \quad \text{for} \vee i$$

$$\therefore \beta_1 + 2\gamma_1 P_1 = \lambda_2 P_2 = \lambda_2, \quad \text{and} \quad P_2 = P_D - P_1.$$

Assuming that $\lambda_1 = \lambda_2 = \lambda$, then

$$\beta_1 + 2\gamma_1 P_1 = \beta_2 + 2\gamma(P_D - P_1)$$

$$\Rightarrow \lambda = \frac{\beta_2 - \beta_1}{2(\gamma_1 + \gamma_2)} = \frac{6.01 - 5.93}{2(4.306 - 4.812) \times 10^{-3}}$$

$$= 655 \text{ MW}.$$

But $P_{1\max} = 500$ MW; therefore we must set P_1 to 500 MW. It follows that the incremental cost for each unit must then be recalculated.

Now, $P_2 = P_D - P_1 = 700 - 500 = 200$ MW, a value within the desired limits of P_2. Therefore, $\lambda_1 = \beta_1 + 2\gamma_1 P_1 = 5.93 + (2)(4.306 \times 10^{-3})(500) = 10.236$ \$ MW h and $\lambda_2 = \beta_2 + 2\gamma_2 P_2 = 6.01 + (2)(4.812 \times 10^{-3})(200) = 7.935$ \$ MW h.

This is an example of a much-simplified iterative process whereby the constraints impose limitations on the desired values of λ_i for each unit.

4.8 Conclusion

This chapter handled practical system problems formulated as constrained optimization problems. The definition of admissible points was presented as those vectors that satisfy the constraints of the problems. It was shown that constrained problems are assumed to have at least one admissible point. The so-called Lagrange multipliers were used as an effective way of dealing with constraints. In Section 4.8.2 theorems on the optimization of constrained functions were presented and necessary and sufficient conditions were defined. In Section 4.8.3 a procedure for optimizing constrained problems was stated in the form of sequential steps. Section 4.8.4 presented some solved problems for illustration and validation purposes. Finally, Section 4.8.5 presented power system application examples such as optimal operation of all thermal system incremental cost-loading.

4.9 Problem Set

PROBLEM 4.9.1

Consider the following problem.

Maximize $x_1^2 - 4x_1x_2 + x_2^2$

Subject to $x_1 + x_2^2 = 1$.

1. Using the Kuhn–Tucker (K–T) conditions, find an optimal solution to the problem.
2. Test for the second-order optimality condition.

Does the problem have a unique optimal solution?

PROBLEM 4.9.2

Consider the following problem:

Maximize $3x_1 - x_2 + x_2^2$

Subject to $x_1 + x_2^2 + x_3 \leq 0$

$-x_1 - 2x_2 + x_3^2 = 0.$

1. Using the Lagrange multiplier technique, write the K–T optimal conditions.
2. Test for the second-order optimality conditions.
3. Argue why the problem is unbounded.

PROBLEM 4.9.3

Find all local minimum and maximum points of the following functions for the four indicated domains. Be sure to list each of the specified points along with the corresponding functional value at the particular point.

Functions

1. $F(x,y) = x^4 + 6x^2y^2 + y^4 - 2x^2 - 2y^2$.
2. $F(x,y) = x^3 - 3x^2 + 4x + y^2$.
3. $F(x,y) = x^4 + y^4$.

Four indicated domains of the permissible constraints set for each of the objective function.

1. $\{(x,y)\colon x^2 + y^2 \leq 1\}$.
2. $\{(x,y)\colon x^2 + y^2 \geq 2\}$.

3. $\{(x,y): |x| \leq 1 \text{ and } |y| \leq 1\}$.
4. $\{(x,y): |x| \leq 2 \text{ and } |y| \leq 2\}$.

PROBLEM 4.9.4

Find the local and global maxima and minima of the function

$$f(x) = f(x_1, x_2, x_3) = x_1^2 + 7x_1 + x_1x_2 + x_2^3 + x_2x_3^2$$
$$|x_1|, |x_2|, |x_3| \leq 4.$$

PROBLEM 4.9.5

Two electrical generators are interconnected to provide total power to meet the load. Each generator's cost is a function of the power output (Figure 4.3). All power costs are expressed on a per unit basis. The total power need is at least 60 units. Formulate a minimum cost design problem and solve it graphically. Verify K–T conditions at the solution points and show gradients of cost and constraint functions on the graph.

PROBLEM 4.9.6

Repeat Example 4.7.3 for a transmission loss equation given by

$$P_L = 0.08P_2.$$

All other data are unchanged.

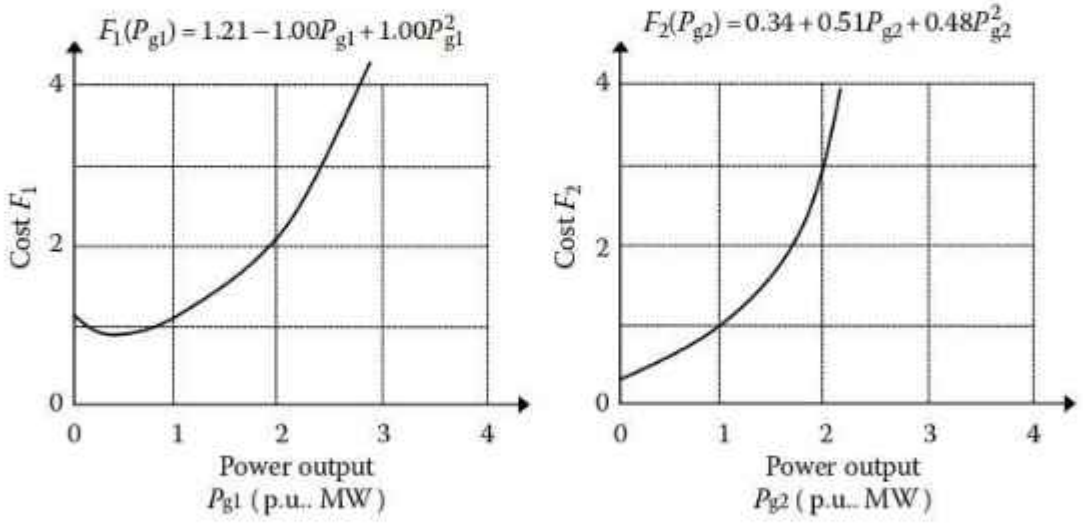


FIGURE 4.3
Power generator cost curves for Problem 4.9.5.

References

1. Bertsekas, D., *Constrained Optimization and Lagrange Multiplier Methods*, Academic Press, New York, 1982; republished by Athena Scientific, 1996.
2. Heterse, M. R., *Optimization Theory, the Finite Dimensional Case*, Wiley, New York, 1975.
3. Hillier, F. S. and Lieberman, G. J., *Introduction to Operations Research*, 4th edn., Holden-Day, San Francisco, CA, 1986.
4. Mulvey, J. M., Vanderbei, R. J., and Zenios, S. A., Robust optimization of large-scale systems, *Operations Research*, 43:264–281, 1995.
5. Luenberger, D. G., *Optimization by Vector Space Methods*, Wiley, New York, 1969.
6. Luenberger, D. E., *Introduction to Linear and Nonlinear Programming*, Addison-Wesley, Reading, MA, 1975.
7. Mokhtar, S., Bazaraa, C., and Shetty, M., *Nonlinear Programming, Theory and Algorithm*, Wiley, New York, 1979.
8. Pierre, D. A., *Optimization Theory with Application*, Wiley, New York, 1969.
9. Potter, W. A., *Modern Foundation of Systems Engineering*, Macmillan, New York, 1966.
10. Zangwill, W. I., Minimizing a function without calculating derivations, *Computer Journal*, 10:293–296, 1967.

5

Linear Programming and Applications

5.1 Introduction

While linear programming (LP) has been a very useful tool developed and used by operation research, power systems, and industry, such as aerospace, for power system, it is fully used in operation and planning of determining resource allocation and value. In recent years, it is a major tool for pricing measured resources [3,8]. It has been of major use in prediction and estimation of data from measurements. The operation research environment and economics have to rely on linear program as a tool for pricing and resources allocation. Due to capability challenge of handling of data new advances to improve computation speed have been introduced namely, interior point method is the application we plan to introduce the concept of LP formulation in matrix form and develop solution requirements based on graphical, simplex (recommended), and interior point method. The application of the methods is numerous in large-scale systems.

Additionally, integer programming (IP) to account for discrete decision variables is introduced in this chapter. The sensitivity method that reduces computational burden by accounting for small perturbation in right-hand side variables will also be found useful to the reader. There are other different techniques, such as calculation for steady-state-based Lagrange technique, which are discussed.

Research papers that apply LP-based optimal power flow (OPF) for minimal losses, cost, etc. use LP to solve optimal operating points of voltage, power as constraints in order to determine the minimized objective values and determine the optimal objectives. The implementation of intermitted resources from distributed generation (DG) is also studied in OPF based on LP [14] provides examples of well-written research work done in the area of LP applications.

The most general description of LP problem [6,7,10–12] is given as the problem of allocating a number m of resources among $1, 2, \ldots, n$ activities in such a way as to maximize the worth from all the activities. The term "linear" refers to the fact that all the mathematical relationships among the decisions (variables) to allocate resources to activities and the various restrictions applicable therein (constraints), as well as the criterion (objective function) are

TABLE 5.1

Notations Commonly Used in LP

	Activity				
Resources	**1**	**2**	**...**	***n***	**Total Resources**
1	a_{11}	a_{12}	...	a_{1n}	b_1
2	a_{21}	a_{22}	...	a_{2n}	b_2
.	.	.	...	.	.
.	.	.	...	.	.
.	.	.	...	.	.
M	a_{m1}	a_{m2}	...	a_{mn}	b_m
ΔP/unit of activity	c_1	c_2	...	c_n	—
Level of activity	x_1	x_2	...	x_n	—

devoid of any nonlinearity. The objective function is some measure of the overall performance of the activities (e.g., cost, profit, net worth, system efficiency, etc.). Standard notation for LP is summarized in Table 5.1.

For activity j, c_j ($j=1,\ldots, n$) is the increase in P (i.e., ΔP), that would result from each unit of increase in x_j (the level of activity j). For resource i, $i=1,\ldots,m$, b_i is the total amount available for allocation to all the activities. The coefficient a_{ij} denotes the amount of resource i consumed by activity j. The set of inputs (a_{ij}, b_i, c_j) constitutes the parameters of the LP model.

5.2 Mathematical Model and Nomenclature in LP

The conventional LP model reduces to the following equation:

$$\text{Maximize } P = c^{\mathrm{T}}x \tag{5.1}$$

Subject to

$$Ax \leq b \tag{5.2}$$

$$x_j \geq 0, \quad \forall j \in \{1,n\}, \tag{5.3}$$

where the following vectors are defined

$$\text{Decision matrix: } x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}}.$$

$$\text{Cost coefficient array: } c^{\mathrm{T}} = [c_1, c_2, \ldots, c_n].$$

$$\text{Constant array: } b = [b_1, b_2, \ldots, b_m]^{\mathrm{T}}.$$

System or state matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

The following are important terminologies used in LP:

Objective functions and constraints. The function $P(x)$ being maximized is called the objective or goal function subject to the restriction sets of constraints given by Equations 5.2 and 5.3. Equation 5.2 represents a restriction set that is often referred to as the functional constraints and Equation 5.3 is termed as the nonnegativity constraints.

Feasible solution and region. Any specification of the variable x_j is called a solution. A feasible solution is one that satisfies all constraints. The feasible region is the collection of all feasible solutions. If the problem does not have any feasible solution, it is called an infeasible problem.

Optimal solution. An optimal solution corresponds to the minimum or maximum value of the objective function. The problem is either one of minimization or maximization depending on the nature of the objective function under consideration, that is, cost and profit, respectively.

Multiplicity in solution. There can be multiple optimal solutions in cases where a number of combinations of the decision variables give the same maximum (or minimum) value.

Unbounded solutions. There may also be unbounded solutions in that the LP problem objective function could be infinitely low or high depending on minimizing or maximizing cases, respectively.

5.2.1 Implicit Assumptions in LP

There are certain assumptions utilized in LP models that are intended to be only idealized representations of real problems. These approximations simplify the real-life problem to make it tractable. Adding too much detail and precision can make the model too unwieldy for useful analysis. All that is needed is that there be a reasonably high correlation between the prediction of the model and what would actually happen in reality.

The four assumptions are as follows:

1. *Proportionality.* This relates to the assumption of constant-cost coefficients c_j irrespective of the level of x_j. Stated alternatively, there can be correlation between the coefficients c_j and x_j. For example, the incremental cost associated with additional units of activity may be

lower, which is referred to as "increasing marginal return" in the theory of economics. The proportionality assumption is needed essentially to avoid nonlinearity.

2. *Additivity.* Apart from the proportionality assumption, the additivity assumptions are needed to avoid cross-product terms among decision variables. This assumption amounts to state that the total contribution from all activities can be obtained by adding individual contributions from respective activities.
3. *Divisibility.* Decision variables can take any fractional value between specified limits. In other words, the variables cannot be restricted to take some discrete values or integer values.
4. *Certainty.* All the parameters in the model are assumed to be known constants with no uncertainty about the values that these parameters may assume.

It is important in most cases to study and analyze the disparities due to these assumptions by checking with more complex models. The more complex alternative models are a result of relaxing the assumptions regarding nonlinear programming models due to relaxing assumption (1) and/or assumption (2), IP models by relaxing assumption (3), and stochastic programming models by ignoring assumption (4). More complex models are obtained by dropping various combinations of assumptions (1)–(4).

5.3 LP Solution Techniques

Various approaches have been developed over the years to solve LP problems. The commonly encountered techniques that have gained wide attention from engineers, mathematicians, and economists are the graphical approach, the simplex method, the revised simplex method, and the tableau approach. In the text, we omit the treatment of the tableau approach as its flexibility in the development of programs for more complex algorithms outdoes its ability to solve large-scale problems [1–5].

5.3.1 Graphical Method

In the LP problem, the optimal solution (if it exists) lies at one of the corner points of the polytope formed by the boundary conditions of the functional and nonnegativity constraints of the problem. In the graphical method, each corner-point solution, which is also a feasible solution, is checked using the objective function. The solution that yields the greatest improvement to the objective value is the optimal solution to the problem. The graphical technique for solving LP problems is demonstrated by the following example.

Consider the following two-variable LP problem.

Maximize $P = 3x_1 + 5x_2$

Subject to

$x_1 \leq 4$

$2x_2 \leq 12$

$3x_1 + 2x_2 \leq 18$

$x_j \geq 0, \quad \forall j \varepsilon \{1,2\}.$

A two-dimensional figure can be constructed corresponding to the two variables x_1 and x_2 (Figure 5.1). The nonnegativity constraints automatically imply that the search is restricted to the positive side of the x_1 and x_2 axes. Next, it should be observed that the feasible solution cannot lie to the right of the line $x_1 = 4$, because of the restriction $x_1 \leq 4$. In the same way, the restrictions $2x_2 \leq 12$, and $3x_1 + 2x_2 \leq 18$ provide the other two cuts to generate the feasible region in Figure 5.1 as indicated by the shaded area OABCD.

The vertices A, B, C, D, and the origin of the polytope are the corner-point solutions to this problem and at least one of such points represents the optimal solution set for x_1 and x_2. We must now select the point in this region that maximizes the values of $P = 3x_1 + 5x_2$. Different lines corresponding to different values of P are drawn to check the point beyond which there is no point of the feasible region, which lies on the line $P = 3x_1 + 5x_2$. This is

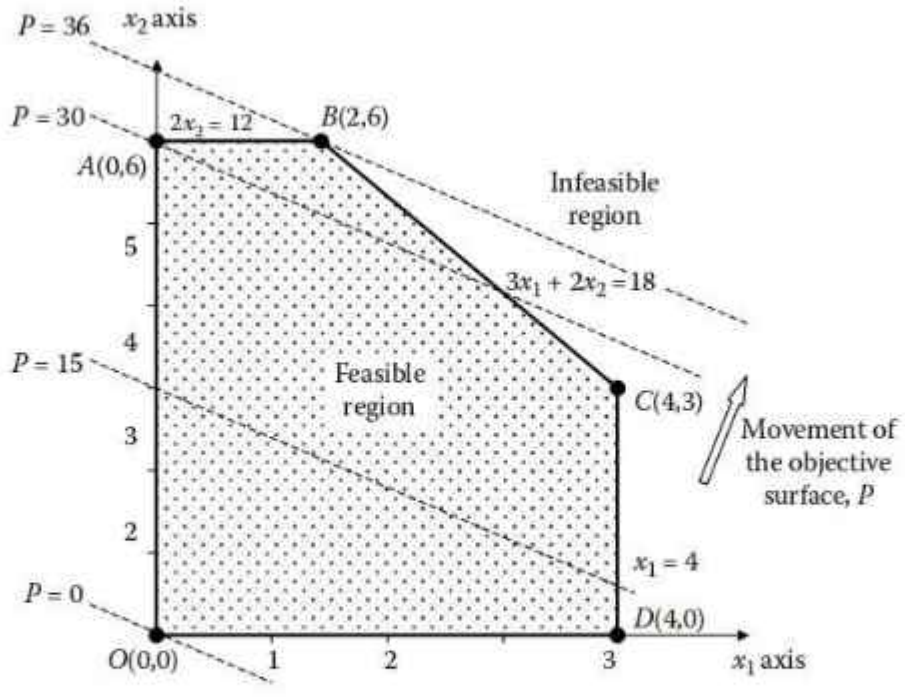


FIGURE 5.1
Graphical representation of the two-dimensional problem.

achieved by drawing a line for $P = 0$ and then progressively increasing it to 15, 30, and so on. It is found that the optimal solution is

$$x_1^* = 2$$
$$x_2^* = 6$$

and the maximum value of P is

$$p^* = 36.$$

This is shown in Figure 5.1 as vertex B of the polytope.

5.3.2 Matrix Approach to LP

The matrix approach is quite effective for the computer-based solution of LP problems. It generally first requires a matrix inversion. The simplex method equipped with an artificially constructed identity matrix for initialization is called the revised simplex method. The revised simplex method and a flowchart for computer programming are discussed with illustrative problems. The IP problem is to be solved as an LP in conduction with Gomory cut, which eliminates the nonintegral optimal solution step by step.

The purpose of LP is to solve the problem formulated in the following standard form:

$$\text{Maximize } p = c^{\mathrm{T}}x \tag{5.4}$$

Subject to

$$Ax = b \tag{5.5}$$

$$\text{and } x \geq 0, \tag{5.6}$$

where

$b_i > 0$, for all $i = 1, 2, \ldots, m$.

$x_j \geq 0$, for all $j = 1, 2, \ldots, n$.

We now define the following arrays in preparation for formulating the LP problem in this chapter:

1. m-vectors

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad x_{\mathrm{B}} = \begin{bmatrix} x_{\mathrm{B}1} \\ x_{\mathrm{B}2} \\ \vdots \\ x_{\mathrm{B}m} \end{bmatrix}.$$

2. n-vectors

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad c = [c_1, c_2, \ldots, c_n].$$

3. $n \times m$ vectors for the case when $n > m$,

$$Z = [z_1, z_2, \ldots, z_{n-m}].$$

4. Matrices

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Now let B represent an $m \times m$ matrix and $\bar{B}$ represent an $m \times (n-m)$ matrix such that $A = [B, \bar{B}]$. Matrix A is now said to be augmented into two components.

The state vector x is a feasible solution (or simply called a solution) if it satisfies the constraint Equations 5.5 and 5.6. It is a nondegenerate basic feasible solution (or simply called a basic solution) if it contains exactly m positive components with all others equal to zero. It is a maximum feasible solution (or simply called a maximum solution) if it is a solution and also maximizes the objective function P.

In practical applications, a given problem may not appear the same as the standard form. For example, the state variable x_1 may be bounded, the constraints may be inequalities, or the objective function may require minimization. In most cases, the following rules can be useful in converting a practical problem to the standard form:

Rule 1: Change of constraints on state variables

The bounded state variable of the following forms can be converted to the inequality form (Equation 5.6):

1. If $x_i \leq d_i$, then add a slack variable x_s to make $x_i + x_s = d_i$ or replace x_i by $d_i - x_s$ where x_s is nonnegative.
2. If $x_i \geq d_i$, then add a slack variable x_s to make $x_i - x_s = d_i$ or replace x_i by $x_s + d_i$.
3. If $-\infty < x_i < \infty$, then replace x_i by $x_s - x_{s+1}$.
4. If $d_i \leq x_i \leq h_i$, then the constraint is equivalent to $0 \leq x_i - d_i \leq h_i - d_i$ and hence x_i is to be replaced by $x_s + d_i$ and $x_s + x_{s+1} = h_i - d_i$ is a new constraint.

Rule 2: Conversion from inequality to equality constraint

We use $(A_x)_i$ to denote the ith row of the vector (Ax) in the following:

1. If $d_i \leq (Ax)_i \leq h_i$, then two slack variables are needed to make $(Ax)_i - x_s = d_i$ and $(Ax)_i + x_{s+1} = h_i$.
2. If $(Ax)_i = b_i \leq 0$, then multiply both sides by -1 when $b_i < 0$, and create a new constraint by adding it to another constraint when $b_i = 0$.

Rule 3: Modification of objective function

1. Minimization of P. Any solution for $\max(-P)$ is the same as that for $\min(P)$, but $\min(P) = -\max(-P)$.
2. Maximization of $|P|$. Find both $\max(P)$ and $\min(P)$ subject to the same constraint and then select the larger of $|\max(P)|$ or $|\min(P)|$.

5.3.3 Simplex Method

The geometric method cannot be extended to large-scale problems for which the number of decision variables can run into several thousands. The simplex method, developed by Dantzig in 1947 [13], and its variants have been widely used to date and a number of commercial solvers have been developed based on it. The first step in setting up the simplex method converts the inequality constraints into equivalent equality constraints by adding slack variables.

Recall the following example:

$$\text{Maximize } P = 3x_1 + 5x_2$$

Subject to

$$2x_2 \leq 12$$

$$3x_1 + 2x_2 \leq 18$$

$$x_j \geq 0, \quad \forall j \in \{1,2\}.$$

This example illustrative problem can be converted into the following equivalent form by adding three slack variables x_3, x_4, and x_5. Thus, we obtain

$$\text{Maximize } P = 3x_1 + 5x_2$$

Subject to

$$x_1 + x_3 = 4$$

$$2x_1 + x_4 = 12$$

$$3x_1 + 2x_2 + x_5 = 18$$

$$x_j \geq 0, \quad \forall j \in \{1,2\}.$$

This is called the augmented form of the problem and the following definitions apply to the newly formulated LP problem.

An augmented solution is a solution for the original variables that has been augmented by the corresponding values of the slack variables.

A basic solution is an augmented corner-point solution.

A basic feasible solution is an augmented corner-point feasible solution.

For example, augmenting the solution (3, 2) in the example yields the augmented solution (3, 2, 1, 8, 5); and the corner-point solution (4, 6) is translated as (4, 6, 0, 0, −6).

The main difference between a basic solution and a corner-point solution is whether the values of the slack variables are included. Because the terms basic solution and basic feasible solution are integral parts of LP techniques, their algebraic properties require further clarification. It should be noted that there are five variables and only three equations in the present problem. This implies that there are two degrees of freedom in solving the system since any two variables can be chosen to be set equal to any arbitrary value in order to solve the three equations for the three remaining variables. The variables that are currently set to zero by the simplex method are called nonbasic variables, and the others are called basic variables. The resulting solution is a basic solution. If all the basic variables are nonnegative, the solution is called a basic feasible solution. Two basic feasible solutions are adjacent if all but one of their nonbasic variables are the same.

The final modification needed to use the simplex method is to convert the objective function itself in the form of a constraint to get

$$P - 3x_1 - 5x_2 = 0.$$

Obviously, no slack variables are needed since it is already in the form of equality. Also, the goal function P can be viewed as a permanent additional basic variable.

The feasibility of achieving a solution for the LP problem as described by Equations 5.4 through 5.6 relies on a set of optimality conditions. In solving the problem, the optimality conditions are forced to be satisfied by means of the simplex method, involving a systematic process of selecting a basis to increase the objective function P until it can no longer be increased. We assume that the problem is nondegenerate; that is, all solutions are basic. The following notations are used in the statement and the theorem in this chapter:

B	m-square and nonsingular matrix formed by m columns of A
x_B	m-vector of which the components are the positive components of a basic solution x corresponding to B; that is, $x_B = B^{-1}b$
C_B	m-row vector formed by m components of C that correspond to x_B; that is, $P = Cx = C_B x_B$
$\bar{C}_B$	$(n-m)$-row vector makes $C = (C_B, \bar{C}_B)$
$\bar{c}_{BK}$	kth component of $\bar{C}_B$
$\bar{a}_k$	kth column of the matrix $\bar{B}$
Z	$(n-m)$-row vector defined by $Z = C_B B^{-1} \bar{B}$
z_k	kth component of Z

THEOREM 5.3.1

The objective function P can be increased if $z_k - \bar{c}_{Bk} < 0$ *for some k, and the components of* x_B *are the positive components of a maximum solution if* $Z - \bar{C}_B \leq 0$.

The theorem gives a condition of optimality that suggests a systematic approach to a maximum solution of the LP problem.

It is assumed in the simplex method that initially there exists a basis, which yields a basic solution. In practical problems, such a basis is hard if not impossible to find, especially for large m. To overcome this difficulty, an identity matrix is created here for the initialization of the simplex method. The simplex method requires that the basis matrix be inverted at each iteration. We note that there is only a one column change between two consecutive bases. This enables us to apply a lemma of matrix inversion to obtain the inverse matrix for the new basis from that of the old one.

Consider the following example:

$$x_1 + x_2 + x_3 + x_4 = b_1$$
$$x_1 + 2x_2 + x_3 = b_2$$
$$2x_1 - x_2 - x_3 = b_3.$$

Introduce artificial variables x_5 and x_6 such that

$$x_1 + x_2 + x_3 + x_4 = b_1$$
$$x_1 + 2x_2 + x_3 + x_5 = b_2$$
$$2x_1 - x_2 - x_3 + x_6 = b_3.$$

Then, the matrix A of Equation 5.5 has the form

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

and, hence, an identity matrix is created by the last three columns. The basic solution corresponding to the basis of the identity matrix is, therefore, $x_4 = b_1$, $x_5 = b_2$, and $x_6 = b_3$. There are no more than m artificial variables because some state or slack variables can be utilized as artificial variables, such as x_4 in this example.

5.3.4 Lemma of Matrix Inversion

The equation

$$[P^{-1} + H_1 Q H_2]^{-1} = P - PH_1[H_2 PH_1 + Q^{-1}]^1 H_2 P \qquad (5.7)$$

is true if the inverses exist. The lemma can be applied generally to the n-square matrix P, m-square matrix Q, $n \times m$ matrix H_1, and $m \times n$ matrix H_2. The integer m is practically less than n.

We make use of the identity to generate a recursive formula of matrix inversion that is useful to LP. To this end, we assume that $p = B^{-1}$, $Q = 1$, $H_1 = \bar{a}_k - a_r$, and $H_2 = J_r^T$, where J_r is the column matrix with rth component equal to one but all others equal to zero. Thus, the following result can be obtained for nonsingular matrix B:

$$\begin{aligned}(a)\ 1 + H_2 B^{-1} H_1 &= 1 + J_r^T B^{-1}(\bar{a}_k - a_r) \\ &= 1 + J_r^T y - J_r^T B^{-1} a_r \\ &= 1 + y_r - J_r^T J_r = y_r\end{aligned}$$

where

$B^{-1}\bar{a}_k = y = [y_1, y_2, \ldots, y_m]^T$

a_r is the rth column of the B matrix

$$\begin{aligned}(b)\ B^{-1} H_1 H_2 B^{-1} &= B^{-1}(\bar{a}_k - a_r) J_r^T B^{-1} \\ &= (y - J_r) J_r^T B^{-1} \\ &= (y_1, y_2, \ldots, y_{r-1}, \ldots, y_m)^T (b_{r1}, b_{r2}, \ldots, b_{rm}).\end{aligned}$$

Substitutions of (a) and (b) into the lemma give

$$(B + H_1 H_2)^{-1} = B^{-1} - \frac{1}{y_r}(y_1, y_2, \ldots, y_{r-1}, \ldots, y_m)^T (b_{r1}, b_{r2}, \ldots, b_{rm}).$$

Let b_{ij} and b'_{ij} be the general elements of the matrices B^{-1} and $(B + H_1 H_2)^{-1}$, respectively. Then, it follows from the above equations that

$$b'_{ij} = b_{ij} - \frac{y_i}{y_r} b_{rj}; \quad \text{for } i \neq r \tag{5.8}$$

$$= \frac{b_{rj}}{y_r}; \quad \text{for } i = r \tag{5.9}$$

As such, the matrix inversion for the new basis can be derived from that of the old one according to Equations 5.8 and 5.9. Now, the simplex method can be broken down into three stages: namely initialization, iterative procedure, and optimality checking.

1. *Initialization step*: The simplex method can start at any corner-point feasible solution; thus a convenient one such as (0, 0) is chosen for the original variables. Therefore the slack variables become the basic variables and the initial basic feasible solution is obtained at (0, 0, 4, 12, 18).

2. *Iterative procedure*: The iterative procedure involves moving from one corner-point solution to an adjacent basic feasible solution. This movement involves converting one nonbasic variable into a basic variable and simultaneously converting a basic variable into a nonbasic one. The former is referred to as the entering variable and the latter as the leaving variable.

The selection of the nonbasic variable to be converted into a basic variable is based on improvement in the objective function P. This can be calculated from the new representation of the objective function in the form of a constraint. The nonbasic variable, which contributes to the largest increment in the P value, is the entering variable. For the problem at hand, the two nonbasic variables x_1 and x_2 add per-unit contribution to the objective function as 3 and 5, respectively (the coefficients c_j), and, hence, x_2 is selected as the entering variable.

The adjacent basic feasible solution is reached when the first of the basic variables reaches a value of zero. Thus, once the entering variable is selected, the leaving variable is not a matter of choice. It has to be the current basic variable whose nonnegativity constraint imposes the smallest upper bound on how much the entering basic variable can be increased. The three candidate leaving variables are x_3, x_4, and x_5. Calculation of the upper bound is illustrated in Table 5.2.

Since x_1 remains a nonbasic variable, $x_1 = 0$ which implies x_3 remains nonnegative irrespective of the value of x_2, $x_4 = 0$, when $x_2 = 6$, and $x_5 = 0$, when $x_2 = 9$, indicate that the lowest upper bound on x_2 is 6, and is determined when $x_4 = 0$. Thus, x_4 is the current leaving variable.

3. *Optimality test*: To determine whether the current basic feasible solution is optimal, the objective function can be rewritten further in terms of the nonbasic variables:

$$P = 30 + 3x_1 - \frac{5}{2}x_4.$$

Increasing the nonbasic variables from zero would result in moving toward one of the two adjacent basic feasible solutions. Because x_1 has a positive coefficient increasing x_1 would lead to an adjacent basic feasible solution that is better than the current basic feasible solution; thus, the current solution is

TABLE 5.2

Calculation of Upper Bound

Basic Variable	Equation	Upper Bound for x
x_3	$x_3 = 4 - x_1$	No limit
x_4	$x_4 = 12 - 2x_2$	$x_2 \leq \frac{12}{2} = 6 \leftarrow \min$
x_5	$x_5 = 18 - 3x_1 - 2x_2$	$x_2 \leq \frac{18}{2} = 9$

not optimal. Speaking generally, the current feasible solution is optimal if and only if all of the nonbasic variables have nonpositive coefficients in the current form of the objective function. The flowchart of the simplex method is shown in Figure 5.2.

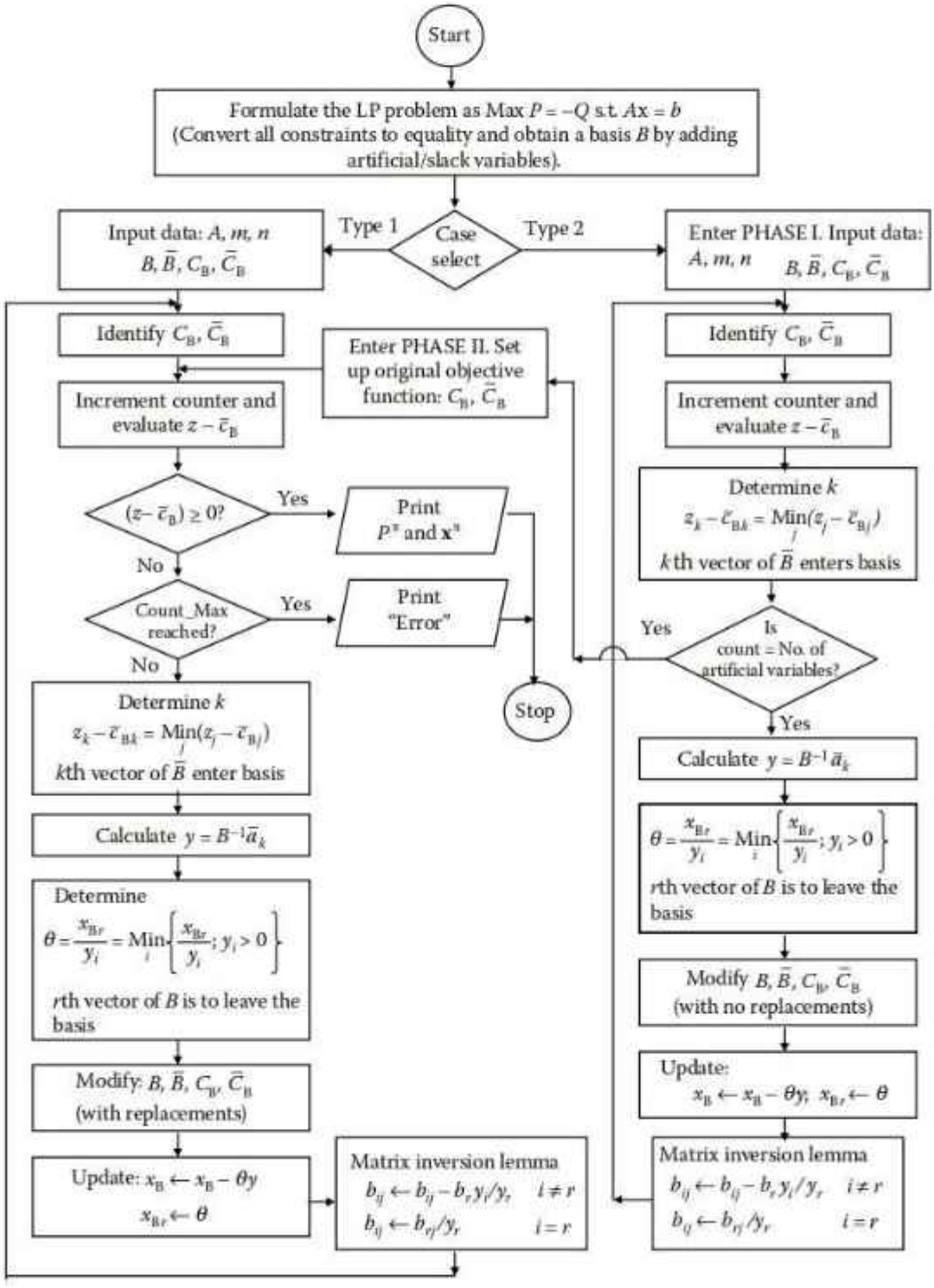


FIGURE 5.2
Flowchart showing the three stages of the simplex method.

5.3.5 Revised Simplex Method

The so-called revised simplex method consists of two phases:

Phase I
This first phase starts with an identity matrix for the basis (an artificial basis) and the maximization of x_0, where

$$x_0 = -(\text{sum of all the artificial variables}).$$

The coefficients of the objective function x_0 are all equal to -1. If the maximized x_0 is less than zero, then there is at least one of the artificial variables that is different from zero and, therefore, no feasible solution exists for the problem. We enter phase II if all the artificial variables are eliminated or $\max(x_0 = 0)$.

Phase II
The aim of phase II is to maximize the goal function P of the problem with a matrix inversion for the basis inherited from phase I. It should be noted that the constraints of both phases are the same although the objective functions are different. Both phase I and phase II can be implemented either by digital computers or by hand calculations in accordance with the flowchart.

Illustrative example

Minimize $Q = 2x_1 + x_2 + x_3$

Subject to the inequality constraints given by

$3x_1 + 5x_2 + 2x_3 \geq 16$

$4x_1 - 2x_2 + x_3 \geq 3$ and $x_i \geq 0$, for all $i \in \{1,2,3\}$

using the

1. Simplex method
2. Revised simplex method

1. Solution using the simplex method
We convert the minimization problem to a maximization one by changing the sign such that

$$P = -2x_1 - x_2 - x_3.$$

Utilizing two slack variables x_4 and x_5 and two artificial variables x_6 and x_7 we obtain the following equations:

$$3x_1 + 5x_2 + 2x_3 - x_4 + x_6 = 16$$

$$4x_1 - 2x_2 + x_3 - x_5 + x_7 = 3.$$

In matrix notation

$$A = [\bar{B} \ : \ B] = \left[\begin{array}{ccccc:cc} 3 & 5 & 2 & -1 & 0 & 1 & 0 \\ 4 & -2 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$x = [x_1, x_2, x_3, x_4, x_5, x_6, x_7]^T$$

$$C = \left[\bar{C}_B \ \vdots \ C_B\right] = [-2 \quad -1 \quad -1 \quad 0 \quad 0 \ \vdots \ 0 \quad 0]$$

and

$$b = [16, 3]^T.$$

Notably, $B = B^{-1} = I$. Thus, $x_B = B^{-1}b = Ib = [16, 3]^T$.

First iteration

$$\begin{aligned} Z - C_B &= C_B B^{-1}\bar{B} - \bar{C}_B = -\bar{C}_B \\ &= [2 \quad 1 \quad 1 \quad 0 \quad 0], \end{aligned}$$

which is definitely positive.

$$\begin{aligned} P = C_B x_B &= C_B B^{-1} = C_B Ib \\ &= [0 \quad 0]\, I \begin{bmatrix} 16 \\ 3 \end{bmatrix} = 0 \end{aligned}$$

and

$$x_B = \begin{bmatrix} 16 \\ 3 \end{bmatrix} = \begin{bmatrix} x_6 \\ x_7 \end{bmatrix}$$

$$x_1 = x_2 = x_3 = 0$$

$\max P = 0 \Rightarrow \min Q = 0.$

NOTE: This solution is infeasible as the second constraint $4x_1 - 2x_2 + x_3 \geq 3$ is not valid here as it is zero. The revised simplex method shows this result.

2. Solution using the revised simplex method

Phase I

$$\max(x_0) = -(x_6 + x_7)$$

$$C_B = [-1, -1]$$

$$\bar{C}_B = [0 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$x_B = B^{-1}b = \begin{bmatrix} 16 \\ 3 \end{bmatrix} = \begin{bmatrix} x_6 \\ x_7 \end{bmatrix}.$$

First iteration

Step 1

$$\begin{aligned} Z - \bar{C}_B &= C_B B^{-1}\bar{B} - \bar{C}_B \\ &= [-1 \quad -1]I\begin{bmatrix} 3 & 5 & 2 & -1 & 0 \\ 4 & -2 & 1 & 0 & -1 \end{bmatrix} \\ &= [-7 \quad -3 \quad -3 \quad 1 \quad 1]. \end{aligned}$$

The minimum of the previous row is -7; then the first vector of $\bar{B}$, x_1 enters the basis $k=1$.

Step 2

Calculate

$$y = B^{-1}\bar{a}_k = I\bar{a}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\theta = x_{Br} = \min_i \left\{ \frac{x_{Bi}}{y_i};\ y_i > 0 \right\}$$

$$= \min\left\{ \frac{16}{3}, \frac{3}{4} \right\} = \frac{3}{4} \quad \text{and} \quad r = 2.$$

Now, $r=2$ implies that the second vector of B is to leave the basis.

Step 3

The new basic solution is

$$x_B = \begin{cases} x_{Br} = \theta, & i = r \\ x_{Bi} - \theta y, & i \neq r \end{cases}$$

$$x_B = \begin{cases} x_1 = \frac{3}{4} \\ x_7 = 3 - (4(3/4)) = \begin{Bmatrix} \frac{3}{4} \\ 0 \end{Bmatrix}. \end{cases}$$

Step 4

Now we calculate the inverse of the new basic

$$B^{-1} = \left[b'_{ij} \right], \quad \text{for } i = r = 2, \quad b'_{ij} = \frac{b_{rj}}{y_r}$$

$$b'_{21} = \frac{b_{21}}{y_2} = 0 \quad \text{and} \quad b'_{22} = \frac{b_{22}}{y_2} = \frac{1}{4}$$

$$\text{for } i \neq r \Rightarrow i \neq 2 \quad b'_{ij} = b_{ij} - \frac{y_i}{y_r} b_{rj}$$

$$b'_{11} = b_{11} - \left(\frac{y_1}{y_2}\right) b_{21} = 1 - 0 = 1$$

$$b_{21} = b_{21} - \left(\frac{y_1}{y_2}\right) b_{22} = 0 - \left(\frac{3}{4}\right)(1) = \frac{-3}{4}$$

$$[\bar{B}] = \begin{bmatrix} 5 & 2 & -1 & 0 \\ -2 & 1 & 0 & -1 \end{bmatrix}$$

$$[\bar{C}_B] = [-1 \quad -1 \quad 0 \quad 0]$$

$$\bar{x}_b = [x_2 \quad x_3 \quad x_4 \quad x_5]$$

$$[B] = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} \Rightarrow [B^{-1}] = \frac{1}{4}\begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix}$$

$$[C_B] = [0 \quad -2] x_b = [x_b \quad x_1] = \left[\tfrac{3}{4} \quad 0\right].$$

Second iteration

Step 1

$$C_B B^{-1} = \frac{1}{4}[0 \quad -2]\begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{4}[0 \quad -2]$$

$$Z - \bar{C}_B = C_B B^{-1}\bar{B} - \bar{C}_B$$

$$= \frac{1}{2}[0 \quad -1]\begin{bmatrix} 5 & 2 & -1 & 0 \\ -2 & 1 & 0 & -1 \end{bmatrix} - \bar{C}_B$$

$$= \frac{1}{2}[2 \quad -1 \quad 0 \quad 1] - [-1 \quad -1 \quad 0 \quad 0]$$

$$= \left[2 \quad \tfrac{1}{2} \quad 0 \quad \tfrac{1}{2}\right],$$

which is positive definite; then

$$\max P = C_B x_B = [0 \quad -2]\begin{bmatrix} \frac{3}{4} \\ 0 \end{bmatrix} = 0$$

and all the artificial variables are not eliminated from the basis. Therefore we cannot enter phase II of the calculations as the function has an infeasible solution as shown in the simplex method.

5.4 Duality in LP

LP problems exhibit an important property known as duality. The original problem is therefore referred to as the primal and there exists a relationship

between the primal of the LP problem and its duality. Now, from Equations 5.1 and 5.2, the general LP problem in its primal form may be expressed as

Maximize $P(x) = c^{\mathrm{T}}x$

Subject to the constraints:

$Ax \leq b$

and

$x_j \geq 0$

$\forall j \in \{1,n\}$,

where
A is an $m \times n$ matrix
x is a column n-vector
c^{T} is a row n-vector
b is a column m-vector

The LP model above may also be written as

$$\text{Maximize } P(x) = \sum_{j=1}^{n} c_j^{\mathrm{T}} x_j \tag{5.10}$$

Subject to the constraints:

$$\sum_{i}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \leq b_i \right) \tag{5.11}$$

and

$x_j \geq 0$

$\forall i \in \{1,m\}$ and $\forall j \in \{1,n\}$.

Notably, the slack variables have not yet been introduced in the inequality constraints.

By the definition of the dual of the primal LP problem expressed as Equations 5.12 through 5.14, we obtain the following model:

$$\text{Minimize } Q(y) = b_{\mathrm{D}} y \tag{5.12}$$

Subject to the constraints:

$$A_{\mathrm{D}} y \geq c_{\mathrm{D}}^{\mathrm{T}} \tag{5.13}$$

and

$$y_j \geq 0 \tag{5.14}$$

$\forall j \in \{1,m\}$,

where

A_D is an $n \times m$ matrix
y is a column m-vector
c_D is a row m-vector
b_D is a column n-vector

Similar to the model for the primal case, the dual LP model above may also be written as

$$\text{Minimize } Q(y) = \sum_{i=1}^{m} b_{D_i} y_j \tag{5.15}$$

Subject to the constraints:

$$\sum_{j}^{n} \left(\sum_{i=1}^{m} a_{ij} y_j \geq c_j \right) \tag{5.16}$$

$$y_i \geq 0$$

$$\forall i \in \{1,m\} \quad \text{and} \quad \forall j \in \{1,n\}.$$

The dual problem can now be solved using any of the previously discussed solution techniques such as the revised simplex method. In addition, the duality model exhibits special properties that can be summarized using the following well-known duality theorems.

THEOREM 5.4.1

The dual of the dual linear programming model is the primal.

THEOREM 5.4.2

The value of the objective function $P(x)$ for any feasible solution of the primal is greater than or equal to the value of the objective function $Q(y)$ for any feasible solution of the dual.

THEOREM 5.4.3

The optimum value of the objective function $P(x)$ of the primal, if it exists, is equal to the optimum value of the objective function $Q(y)$ of the dual.

THEOREM 5.4.4

The primal has an unbounded optimum if and only if the dual has no feasible solution. The converse is also true.

Finally, we make note of the fact that the duality principle, when applied correctly, can reduce the computation needed to solve multidimensional problems. This is attributed to the fact that the large number of constraints are now modeled as the objective function and the old objective with smaller dimension has become the new constraints. Also, the duality concept can be applied to problems that are associated with the inverse of the A matrix in the primal model of the LP problem.

5.5 Khun–Tucker Conditions in LP

In this section, selected formulations for computing the Lagrange multipliers in LP are summarized using the standard notations. Here, we investigate Min/Max LP problems with inequality and equality constraints. In order to determine the Lagrange multipliers associated with each case, the concepts of duality, Khun–Tucker (K–T) conditions, and Extended K–T (EKT) conditions are applied.

In general, the solution of the dual LP problem results in the computation of the desired Lagrange vectors for any equality constraints that are present. Primal LP with both equality constraints ($A_1x = b_1$) and inequality constraints ($A_2x \leq b_2$) must be transformed into a purely equality using nonnegative slack variables. This is because the dual of mixed equality and inequality constraints does not exist.

A brief summary of common cases of LP and the corresponding duality conditions now give rise to the appropriate use of complementary slackness and are presented next. Complementary slackness is a term used to describe the orthogonal nature of the dual and primal shadow prices or Lagrange vectors in LP. Also, a simple example is used to demonstrate how Lagrange multipliers for LP problems are determined.

5.5.1 Case 1: LP and KKT Conditions for Problems with Equality Constraints

Recall the formulation

Min c^Tx s.t.

$$Ax \geq b$$

$$x \geq 0,$$

where
c^T and b are n- and m-vectors, respectively
A is an $m \times n$ matrix

The Karush–Khun–Tucker (KKT) conditions requiring optimal solutions $[x^*, \lambda^*, \mu^*]^T$ are

$$Ax \geq b, \quad x \geq 0$$

$$A^T\lambda + \mu = c, \quad \lambda \geq 0, \quad \mu \geq 0$$

$$\lambda^T(Ax - b) = 0, \quad \mu^T x \geq 0,$$

Condition 1: Candidate point must be feasible (primal feasibility of x).

Condition 2: Dual feasibility of λ and μ must be met.

Condition 3: Complementary slackness conditions must be satisfied.

In these conditions for optimality in LP, λ is the vector of Lagrange multipliers associated with $Ax \geq b$ and μ is the vector of Lagrange multipliers associated with $x \geq 0$.

Illustrative example
Compute the shadow prices or Lagrange multipliers for the problem given by

$$\text{Min } f(x) = -2x_1 - 4x_2$$

subject to

$$x_1 - 3x_2 \geq -3$$

$$-2x_1 - 2x_2 \geq -5$$

$$x_1, x_2 \geq 0.$$

Step 1: Determine $\mathbf{c}^T$, $\mathbf{A}$, and $\mathbf{b}$ that correspond to the standard LP formulation Min $c^T x$ s.t. $Ax \geq b$ with $x \geq 0$.
Therefore

$$c^T = [c_1, c_2]^T = [-2, -4]^T$$

$$b = [b_1, b_2]^T = [-3, -5]^T$$

$$x = [x_1, x_2]^T$$

$$A = \begin{bmatrix} +1 & -3 \\ -2 & -2 \end{bmatrix}.$$

Step 2: Solve the LP problem using the simplex or revised simplex method. Solution yields primal solution and objective values of

$$x^* = [x_1^*, x_2^*]^T = [1.12, 1.38]^T \quad \text{and} \quad f^* = -7.75$$

Step 3: Apply KKT

KKT #1: Primal feasibility was satisfied since

$$Ax^* = \begin{bmatrix} +1 & -3 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 1.12 \\ 1.38 \end{bmatrix} \cong \begin{bmatrix} -3.03 \\ -5.00 \end{bmatrix} = b.$$

KKT #2: Dual feasibility to compute λ and μ.

$$\text{Recall } A^T\lambda + \mu = c, \quad \lambda \geq 0, \quad \mu \geq 0$$

$$\Rightarrow \begin{bmatrix} +1 & -3 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}.$$

KKT #3: Complementary slackness conditions

$$\lambda^T(Ax - b) = 0, \quad \mu^T x \geq 0,$$

But since $x^* \neq 0$, then from $\mu^T x \geq 0$, this implies that $\mu^T = 0$.

Therefore

$$\begin{bmatrix} +1 & -3 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} +1 & -3 \\ -2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 0.50 \\ 0.75 \end{bmatrix}.$$

Therefore, the desired shadow prices or Lagrange multipliers for this problem are $\lambda_1 = 0.50$ and $\lambda_2 = 0.75$.

5.5.2 Case 2: KKT Applied to the Dual LP Problem

This approach is computationally efficient as the Lagrange multipliers are the direct solutions of the Dual LP problem. Furthermore, KKT allows us to compute the optimal value of x without necessarily re-solving the primal LP problem.

Now, by the duality transformation, we have

Primal LP

Min $c^T x$ s.t. $Ax \geq b, x \geq 0$

Dual LP

Max $b^T\lambda$ s.t. $A^T\lambda \leq c, \lambda \geq 0$.

The KKT conditions requiring optimal solutions $[x^*, \lambda^*, \mu^*]^T$ for the Dual problem are given as

Condition 1: Dual feasibility of λ and μ must be met.

$$A^T\lambda \leq c, \quad \lambda \geq 0.$$

Condition 2: Candidate point must be feasible (primal feasibility of x).

$$Ax + \mu = b, \quad \lambda \geq 0, \quad \mu \geq 0.$$

Condition 3: Complementary slackness conditions must be satisfied.

$$x^T(A^T\lambda - c) = 0, \quad \mu^T x = 0.$$

Illustrative example
Recall the problem

$$\text{Min } f(x) = -2x_1 - 4x_2$$

subject to

$$x_1 - 3x_2 \geq -3$$

$$-2x_1 - 2x_2 \geq -5$$

$$x_1, x_2 \geq 0.$$

Solution steps are as follows:

Step 1: Transform the problem to its dual form

$$\text{Max } f_D(x) = b^T\lambda = [-3, \ -5]\begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix} \text{ subject to}$$

$$A^T\lambda \leq c \text{ and } \lambda \geq 0 \text{ such that}$$

$$\begin{bmatrix}+1 & -2 \\ -3 & -2\end{bmatrix}\begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix} \leq \begin{bmatrix}-2 \\ -4\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix} \geq \begin{bmatrix}0 \\ 0\end{bmatrix}.$$

Step 2: Solve the LP problem using the simplex or revised simplex method. Solution yields candidate dual solution and objective values of

$$\lambda^* = [\lambda_1^*, \lambda_2^*]^T = [0.50, 0.75]^T \quad \text{and} \quad \text{Max } f_D^* = -7.75.$$

Step 3: Apply KKT

KKT #1: Dual feasibility is established since

$$A^{\mathrm{T}}\lambda^* = \begin{bmatrix} +1 & -2 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 0.50 \\ 1.25 \end{bmatrix} = \begin{bmatrix} -2.0 \\ -4.0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Therefore $A^{\mathrm{T}}\lambda^* \leq c$ holds true.

KKT #2: Primal feasibility
Recall $Ax + \mu = b$, $\mu \geq 0$

$$\Rightarrow \begin{bmatrix} +1 & -3 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}.$$

KKT #3: Complementary slackness conditions

$$x^{\mathrm{T}}(A^{\mathrm{T}}\lambda - c) = 0, \quad x^{\mathrm{T}}\mu = 0,$$

But since $A^{\mathrm{T}}\lambda - c = 0$, then x^{T} and hence $\mu^{\mathrm{T}} = 0$.

$$\text{Therefore } \begin{bmatrix} +1 & -3 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} +1 & -3 \\ -2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1.125 \\ 1.375 \end{bmatrix}.$$

This is the optimal solution, x^* and the minimal value function is $\text{Min } f(x) = f^* = c^{\mathrm{T}}x^* = [-2, -4]\begin{bmatrix} 1.125 \\ 1.375 \end{bmatrix} = -7.75$, which is $\text{Max } f_{\text{Dual}}$.

5.5.3 Case 3: KKT Applied to LP Problems with Equality Constraints

Primal LP
Min $c^{\mathrm{T}}x$ s.t. $Ax = b$ with $x \geq 0$. The primal KKT conditions requiring optimal solutions x^*, λ^*, and μ^* are

Condition 1: Candidate point must be feasible (primal feasibility of x).

$$Ax = b, \quad x \geq 0.$$

Condition 2: Dual feasibility of λ and μ must be met.

$$A^{\mathrm{T}}\lambda + \mu = c, \quad \lambda \text{ is unrestricted}, \quad \mu \geq 0.$$

Condition 3: Complementary slackness conditions must be satisfied.

$$\lambda^{\mathrm{T}}(Ax - b) = 0, \quad \mu^{\mathrm{T}}x \geq 0.$$

Dual LP
Max $b^T\lambda$ s.t. $A^T\lambda \leq c$ with λ is unrestricted. The dual KKT conditions requiring optimal solutions x^*, λ^*, and μ^* are

Condition 1: Dual feasibility of λ and μ must be met.

$$A^T\lambda + \mu = c, \quad \lambda \text{ is unrestricted}, \quad \text{and } \mu \geq 0.$$

Condition 2: Candidate point must be feasible (primal feasibility of x).

$$Ax = b, \quad x \geq 0.$$

Condition 3: Complementary Slackness conditions must be satisfied.

$$x^T(A^T\lambda - c + \mu) = 0, \quad \mu^T x = 0.$$

Primal LP
Min $c^T x$ s.t. $Ax \leq b$ with x is unrestricted. The primal KKT conditions requiring optimal solutions x^*, λ^*, and μ^* are

Condition 1: Primal feasibility

$$Ax \leq b, \quad x \text{ is unrestricted.}$$

Condition 2: Dual feasibility

$$A^T\lambda + \mu = c, \quad \lambda \geq 0, \quad \mu \geq 0.$$

Condition 3: Complementary Slackness

$$\lambda^T(Ax - b) = 0, \quad \mu^T x \geq 0.$$

Dual LP
Max $b^T\lambda$ s.t. $A^T\lambda = c$ and $\lambda \geq 0$. The dual KKT conditions requiring optimal solutions x^*, λ^*, and μ^* are

Condition 1: Dual feasibility

$$A^T\lambda = c, \quad \lambda \geq 0.$$

Condition 2: Primal feasibility

$$Ax + \mu = b, \quad x \text{ is unrestricted} \quad \text{and } \mu \geq 0.$$

Condition 3: Complementary slackness

$$x^T(A^T\lambda - c) = 0, \quad \mu^T x = 0.$$

5.6 Mixed-Integer Programming

Integer and mixed-IP (MIP) problems are special classes of LP where all or some of the decision variables are restricted to integer values. There are many practical examples where the divisibility assumption in LP needs to be dropped and some of the variables can take up only discrete values. However, even greater importance can be attributed to problems where the discrete values are restricted to zero and one only, that is, "yes" or "no" decisions or binary decision variables. In fact, in many instances, MIP problems can be reformulated to have only binary decision variables that are easier to handle. The occurrence of binary variables may be due to a variety of decision requirements, the most common among which are the following.

1. *ON/OFF decisions*: The most common type of binary decision falls into this category for engineering optimization problems. This decision variable can also have alternative representation of GO/NO GO, BUILD/NOT BUILD, OR SCHEDULE/NOT SCHEDULE, and so on, depending on the specific application under consideration in short, medium, and long-term planning contexts.
2. *Logical either-or/and constraints*: Binary variables can also indirectly handle mutual inclusive or exclusive restrictions. For example, there might be cases where a choice can be made between two constraints, so that only one must hold. Also, there could be cases where process B must be selected if process A has already been selected.
3. *K out of N constraints must hold*: Consider the case where the overall model includes a set of N possible constraints such that only some K of these constraints must hold (assuming $K < N$). Part of the optimization task is to choose which combination of K constraints permits the objective function to reach its best possible value. In fact, this is nothing but a generalization of the either-or constraints, and can handle a variety of problems.
4. *Function with N-possible values*: In many real-life problems, the functions do not have smooth continuous properties, but can take up only a few discrete values. For example, consider the following case

$$f(x_1, \ldots, x_n) = d_1 \quad \text{or} \quad d_2, \ldots, d_N. \tag{5.17}$$

The equivalent IP formulation would be

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{N} d_i y_i \tag{5.18}$$

$$\sum_{i=1}^{N} y_i = 1 \tag{5.19}$$

and Y_i = binary (0 or 1) for $i = 1, \ldots, N$.

5. *Fixed-charge problem*: In most problems, it is common to incur a fixed-cost/set-up charge when undertaking a new activity. In a process-engineering context, it might be related to the set-up cost for the production facility to initiate a run. A typical power system example is the start-up cost of a thermal-generating unit. This fixed charge is often independent of the length or level of the activity and, hence, cannot be approximated by allocating it to the (continuous) level of activity variables.

Mathematically, the total cost comprising fixed and variable charges can be expressed as

$$f_j(x_j) = \begin{cases} K_j + C_j x_j & \text{if } x_j > 0 \\ 0 & \text{if } x_j = 0. \end{cases} \tag{5.20}$$

The MIP transformation would look like

$$\text{Minimize } Z = \sum_{j=1}^{n} (C_j x_j + K_j y_j), \tag{5.21}$$

where

$$y_j = \begin{cases} 1 & \text{if } x_j > 0 \\ 0 & \text{if } x_j = 0. \end{cases}$$

Pure integer or MIP problems pose a great computational challenge. While there exist highly efficient LP techniques to enumerate the basic LP problem at each possible combination of the discrete variables (nodes), the problem lies in the astronomically large number of combinations to be enumerated. If there are N discrete variables, the total number of combinations becomes 2^N! The simplest procedure one can think of for solving an integer or MIP problem is to solve the LP relaxation of the problem (i.e., allowing the discrete variables to take continuous value so that the MIP reduces to non-linear programming) and then rounding the noninteger values to the closest integer solution. There are, however, major pitfalls:

1. Resulting integer solution may not be feasible in the first place.
2. Even if the rounding leads to a feasible solution, it may, in fact, be far from the optimal solution.

Algorithmic development for handling large-scale integer or MIP problems continues to be an area of active research. There have been exciting

algorithmic advances during the middle and late 1980s. The most popular method to date has been the branch-and-bound technique and related ideas to implicitly enumerate the feasible integer solutions.

5.6.1 Branch-and-Bound Technique for Binary Integer Programming Problems

The basic philosophy in the branch-and-bound procedure is to divide the overall problem into smaller and smaller subproblems and enumerate them in a logical sequence. The division procedure is called branching and the subsequent enumeration is done by bounding to check how good the best solution in the subset can be, and then discarding the subset if its bound indicates that it cannot possibly contain an optimal solution for the original problem.

The general structure of the MIP problem is

$$\text{Maximize } P(x) = \sum_{j=1}^{n} c_j x_i. \tag{5.22}$$

Subject to the constraints:

$$\sum_{i=1}^{m} \left(\sum_{i=1}^{n} a_{ij} x_j \leq b_i \right) \tag{5.23}$$

and

$$x_j \geq 0 \tag{5.24}$$

$$\forall j \in \{1,n\},$$

and x_j is an integer $\forall i \in \{1,I\}$.

Assume for simplicity of notation that the first I variables are the integer decision variables.

Before a formal description of the branch-and-bound procedure is given, the basic procedures of branching, bounding, and fathoming are illustrated using a simple numerical example.

Illustrative example

Consider the following IP problem:

$$\text{Maximize } P = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to

$$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 0$$
$$x_3 + x_4 \leq 1$$
$$-x_1 + x_3 \leq 0$$
$$-x_2 + x_4 \leq 0$$
$$\forall x_j \in (0,1).$$

This is a pure integer problem and except for its small size is typical of many of the practical decision-making problems.

Branching

Branching involves developing subproblems by fixing the binary variables at 0 or 1. For example, branching on x_1 for the example problem gives

Subproblem 1. $(x_1 = 0)$

Maximize $P = 5x_2 + 6x_3 + 4x_4$

Subject to

$$3x_2 + 5x_3 + 2x_4 \leq 10$$
$$x_3 + x_4 \leq 1$$
$$x_3 \leq 0$$
$$-x_2 + x_4 \leq 0.$$

Subproblem 2. $(x_1 = 1)$

Maximize $P = 9 + 5x_2 + 6x_3 + 4x_4$

Subject to

$$3x_2 + 5x_3 + 2x_4 \leq 4$$
$$x_3 + x_4 \leq 1$$
$$x_3 \leq 1$$
$$-x_2 + x_4 \leq 0.$$

The procedure may be repeated at each of the two subproblem nodes by fixing additional variables such as x_2, x_3, and x_4. Thus, a tree structure can be formulated by adding branches at each iteration, which is referred to as the solution tree. The variable used to do this branching at any iteration by assigning values to the variable is called the branching variable.

Bounding

For each of the subproblems, a bound can be obtained to determine how good its best feasible solution can be. Consider first the relaxed LP formulation for the overall problem, which yields the following solution:

$$(x_1, x_2, x_3, x_4) = \left(\frac{5}{6}, 1, 0, 1\right)$$

with $P = 16\frac{1}{2}$.

Therefore, $P \leq 16.5$ for all feasible solutions for the original problem. This bound can be rounded off to 16, because all coefficients in the objective function are integers; hence, all integer solutions must have an integer value for P. The bound for whole problem is $P \leq 16$.

In the same way, the bounds for the two subproblems are obtained:

Subproblem 1. $(x_1, x_2, x_3, x_4) = (0, 1, 0, 1)$
with $P = 9$.

Subproblem 2. $(x_1, x_2, x_3, x_4) = (1, \frac{4}{5}, 0, \frac{4}{5})$
with $P = 16.2$.

Therefore, the resulting bounds are

Subproblem 1. $P \leq 9$,
Subproblem 2. $P \leq 16$.

Fathoming

If a subproblem has a feasible solution, it should be stored as the first incumbent (the best feasible solution found so far) for the whole problem along with its value of P. This value is denoted P^*, which is the current incumbent for P.

A subproblem is said to be fathomed, that is, dismissed from further consideration, if

Test 1. Its bound is less than or equal to P.

Test 2. Its LP relaxation has no feasible solutions.

Test 3. Optimal solution for its LP relaxation is an integer; if this solution is better than the incumbent, it becomes the new incumbent, and the test is reapplied to all unfathomed subproblems with the new larger P^*.

Optimality Test

The iterative procedure is halted when there are no remaining subproblems. At this stage, the current incumbent for P is the optimal solution. Otherwise, we return to perform one more iteration. The solution tree of the current example is provided in Figure 5.3.

The markings $F(1)$, $F(2)$, and $F(3)$ on Figure 5.3 indicate that the node has been fathomed by Tests 1, 2, and 3.

For the general branch-and-bound approach in MIP problems, some deviations are necessary to improve the efficiency of the algorithm. These include

1. *Choice of branching variable*: The variables, which have a noninteger solution in the LP relaxation, are selected for branching.
2. *Values assigned to the branching variable for creating subproblems*: Create just two new subproblems by specifying two ranges of values for the variable.

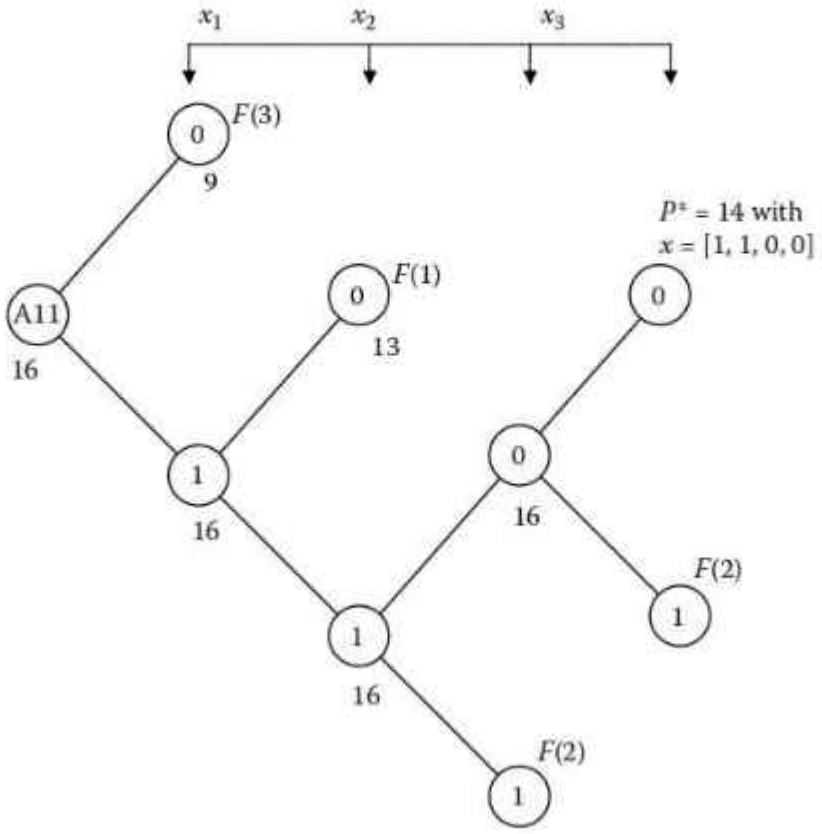


FIGURE 5.3
Solution tree diagram for the IP problem.

3. *Bounding step*: The bound of P is the optimal value of P itself (without rounding) in the LP relaxation.
4. *Fathoming test*: Only the integer decision variables need to be checked for integer solution to decide the fathoming node.

The branch-and-bound procedure is summarized in the following important steps:

Step 1. Initialization: Set $P^* = -\infty$. Apply the bounding step, fathoming step, and optimality test described below to the whole problem. If not fathomed, classify this problem as the one "remaining" subproblem for performing the first full iteration below.

Step 2. Branching: Among the unfathomed subproblems, select the one that was created most recently (breaking ties according to which has the larger bound). Among the integer restricted variables that have a noninteger value in the optimal solution for the LP relaxation of the subproblem, choose the first one in the natural ordering of the variables to be the branching variable. Let x_j be this variable and x_j^* its value in this solution. Branch from the node for the subproblem to create two new subproblems by adding the respective constraints, $x_j \leq [x_j^*]$ and $x_j \geq [x_j^*] + 1$, where $[x_j^*]$ = greatest integer $\leq x_j^*$.

Step 3. Bounding: For each new subproblem, obtain its bound by applying the simplex method to its LP relaxation and using the value of the P for the resulting optimal solution.

Step 4. Fathoming and optimality test: For each new subproblem, apply the three fathoming tests and discard those subproblems that are fathomed by any of the tests.

These are the fundamental steps in the branch-and-bound technique that is applicable to a wide range of MIP problems. For each subproblems that is created, the LP algorithm can be applied in the constrained problem in its pure linear form. We now turn our attention to sensitivity methods in LP.

5.7 Sensitivity Methods for Postoptimization in LP

In many applications, both nonpower and power system types, we often encounter practical problems. In one instance, we seek the optimal solution and in another, we wish to know what happens when one or more variables are changed. In order to save computational effort, it is desirable not to resolve the problem if small perturbations are made to the variables. Sensitivity analysis is the study used to compute such solutions [7,9,10].

Now, recall the LP problem given by the following model as shown in Equations 5.10 and 5.11:

$$\text{Maximize } P(x) = \sum_{j=1}^{n} c_j^{\mathrm{T}} x_j \tag{5.25}$$

Subject to the constraints:

$$\sum_{i}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \leq b_i \right) \tag{5.26}$$

and

$$x_j \geq 0 \tag{5.27}$$

$$\forall i \in \{1,m\} \quad \text{and} \quad \forall j \in \{1,n\}.$$

Here, we observe that changes in the system can be attributed to the modifications such as

1. Perturbation in the parameters, b_i
2. Perturbation in the cost coefficients, c_j
3. Perturbation in the coefficient, a_{ij}

4. Injection of new constraints
5. Injection of new variables

We discuss the effect of each case as it applies to sensitivity analysis and further expound on the first class in more detail in a subsequent section.

5.7.1 Case 1: Perturbation in the Parameters b_1

Let the optimal basis solution for the problem in its primal form be

$$x_B = [x_1, x_2, \ldots, x_m]^T, \tag{5.28}$$

where

$$x_b \geq 0.$$

Since the nonbasic variables are zero, then we can write

$$Bx_B = b \tag{5.29}$$

$$\Rightarrow x_B = B^{-1}b, \tag{5.30}$$

where B is an m-square and nonsingular matrix formed by m columns of A. Let b change to $b + \Delta b$, where $\Delta b = [\Delta b_1, \Delta b_2, \ldots, \Delta b_m]'$, and everything else in the problem remains the same. Then

$$x_B + \Delta x_B = B^{-1}(b + \Delta b), \tag{5.31}$$

given the new values $x_B + \Delta x_B$ of the variables that were the original optimal basic variables.

If $B^{-1}(b + \Delta b) \geq 0$, then the variables continue to be basic feasible. They would also continue to be optimal if the relative cost coefficients given by $C_B B^{-1}\bar{B} - \bar{C}_B$ continued to be nonnegative; that is

$$Z - \bar{C}_B = C_B B^{-1}\bar{B} - \bar{C}_B \geq 0. \tag{5.32}$$

The optimum value of $P(x)$ changes can be calculated with the new values of the variables given by $x_B + \Delta x_B$, or by using the following equation,

$$P(x) = C_B B^{-1}(b + \Delta b). \tag{5.33}$$

5.7.2 Case 2: Perturbation in the Cost Coefficients C_j

If C_j are changed to C_j', everything else in the problem remaining the same, the relative cost coefficients are given by

$$C_B' B^{-1}\bar{B} - \bar{C}_B'. \tag{5.34}$$

These may not all be nonnegative. For some j, some of them are negative. This would mean that the basic feasible solution that was optimal for C_j is not optimal for C_j'. So from this point onwards further iterations may be done with new values C_j' to obtain a new optimal solution.

If, however, C_j' are such that $Z - \bar{C}_B' = C_B' B^{-1} \bar{B} - \bar{C}_B' \geq 0$, then the original optimal basis still remains optimal, and the value of the optimal basic variables also remains unchanged. The optimum value of $P(x)$ is given by

$$P(x) = C_B' B^{-1} b. \tag{5.35}$$

5.7.3 Case 3: Perturbation in the Coefficient a_{ij}

If the changes are in a_{ik}, where x_k is the nonbasic variable of the optimal solution, then we get:

$$C_B' B^{-1} \bar{B}_{(ik)} - \bar{C}_B', \tag{5.36}$$

where $\bar{B}_{(ik)}$ means that the value of a_{ik} in $\bar{B}$ is changed.

If $C_B' B^{-1} \bar{B}_{(ik)} - \bar{C}_B' \geq 0$, then the original optimal basis still remains optimal. If not, further iterations with the new values of $C_B'^{B-1} \bar{B}_{(ik)} - \bar{C}_B'$ and a_{ik} may be done.

5.7.4 Case 4: Injection of New Constraints

Generally, if the original optimal solution satisfies the new constraints that are added to the system, then that solution will still be an optimal solution. However, if some of the injected constraints are violated by the original optimal solution, then the problem must be solved taking into account the new constraints to the system. The new initial point may constitute the old basic variables of the original optimal solution along with one additional basic variable that is associated with each added constraint.

5.7.5 Case 5: Injection of New Variables

Since the number of constraints remains the same, the number of basic variables remains the same. Therefore, the original optimal solution along with zero values of the new variables would result in a basic feasible solution for the new problem. That solution would remain optimal if the newly introduced cost coefficients corresponding to them are nonnegative.

5.7.6 Sensitivity Analysis Solution Technique for Changes in Parameters b_i

We consider the well-known LP problem defined by

$$\min (P = c^T x: \text{ s.t. } Ax = b;\ x \geq 0), \tag{5.37}$$

where
c and x are n-vectors
b is an m-vector

It is assumed that for a given b, the LP problem has been solved with an optimal basis that yields an optimal solution which may be nondegenerate or degenerate. Then the vector b is subject to change with an increment Δb. The postoptimization problem is to further optimize P with respect to Δb under the condition that the optimal basis remains unchanged.

Now let B and X_B be the optimal basis and the associated optimal solution of Equation 5.37. Then it is clear from Equation 5.37 that both P and X_B change as b does. We define the rate of change of P with respect to b as the sensitivity denoted by

$$S_b^P = \frac{\partial P}{\partial b} \text{ (a row } m\text{-vector)}. \tag{5.38}$$

Thus, the new objective function becomes

$$P + \Delta P = P + \frac{\partial P}{\partial b}\Delta b \tag{5.39}$$

$$\Delta P = S_b^P \Delta b. \tag{5.40}$$

The new optimal solution is

$$x'_B = B^{-1}(b + \Delta b) = x_B + H\Delta b \geq 0, \tag{5.41}$$

where $H = B^{-1} = [h_{ij}]$.

In practical applications, only some components of b are subject to change and the changes are usually bounded. If J is the index set that contains j for which b_j changes with increment Δb_j, then

$$f_j \leq \Delta b_j \leq g_j. \tag{5.42}$$

In order to ensure that $\Delta b_j = 0$ is feasible, we impose the condition

$$f_j \leq 0 \leq g_j, \quad \text{for } j \in J. \tag{5.43}$$

The trivial condition does not affect the practicality of the postoptimization, yet it guarantees a solution. It can be shown that the sensitivity S_b^P is the dual solution of Equation 5.37. For this reason, we use the conventional notation

$$y^T = S_b^P, \tag{5.44}$$

where y is the dual solution and a column vector.

The problem is to minimize ΔP with respect to Δb by keeping B unchanged. That is,

$$\text{Minimize } \Delta P = y^{\mathrm{T}}\Delta b = \sum_j y_j \Delta b_j \tag{5.45}$$

Subject to

$$x_{\mathrm{B}i} + h_{ij}\Delta b_j \geq 0 \tag{5.46}$$

and

$$f_i \leq \Delta b_j \leq g_j \tag{5.47}$$

where $j \in J$ and $i = 1, 2, \ldots, m$.

5.7.6.1 Solution Methodology

We wish to utilize the solution of Equation 5.37 to find the sensitivity. Let C_{B} be associated with the optimal basis B and solution X_{B}. That is,

$$P = C_{\mathrm{B}}^{\mathrm{T}} X_{\mathrm{B}} \tag{5.48}$$

and

$$BX_{\mathrm{B}} = b. \tag{5.49}$$

A new vector is defined here to satisfy

$$B^{\mathrm{T}} y = c_{\mathrm{B}}. \tag{5.50}$$

Such an m-vector y is unique since B is nonsingular. In view of Equations 5.49 and 5.50, we know that both P and X_{B} change if the vector b changes in order to maintain the optimality. But B may or may not change due to the insensitive nature of B to b. We consider here the case for which B remains unchanged when b changes. Thus, the vector y of Equation 5.41 is constant when b changes. It follows from Equations 5.48 through 5.50 that

$$\begin{aligned} S_{\mathrm{b}}^{\mathrm{P}} &= \frac{\partial\left(C_{\mathrm{B}}^{\mathrm{T}} X_{\mathrm{B}}\right)}{\partial b} = C_{\mathrm{B}}^{\mathrm{T}} \frac{\partial X_{\mathrm{B}}}{\partial b} = y^{\mathrm{T}} B \frac{\partial X_{\mathrm{B}}}{\partial b} \\ &= y^{\mathrm{T}} \frac{\partial (BX_{\mathrm{B}})}{\partial b} = y^{\mathrm{T}} \frac{\partial b}{\partial b} = y^{\mathrm{T}} I = y^{\mathrm{T}}. \end{aligned} \tag{5.51}$$

This result shows that the sensitivity is indeed the y that satisfies Equation 5.50. Importantly, it is a constant and, hence, ΔP as given by Equation 5.40 is exact without any approximation. Since y is also the dual solution of Equation 5.37 for the nondegenerate case, it can be obtained together with B.

We do not impose the condition of nondegeneracy but only require an optimal basis B which may yield a degenerate ($X_B \geq 0$) or nondegenerate ($X_B > 0$) solution. In any case, y can be found from Equation 5.50 by B^{-1} which is also needed in the algorithm.

It is intended to solve the problem by changing one component of Δb at a time. By looking at the sign of y_j, one may choose a feasible Δb_j in such a way that

$$y_j \Delta b_j = -|y_i \Delta b_j| \leq 0, \tag{5.52}$$

which is equal to zero only when $\Delta b_j = 0$.

A process is employed to change Δb_j step by step with the resetting of necessary quantities. In the process, $j \in J$ advances from the first one to the last and then back to the first, and so on, until all $\Delta b_j = 0$ (steady state). Since ΔP is decreasing from one step to another unless $\Delta b_j = 0$, the steady state is reachable if a solution exists.

The method can be implemented by using a series of logical decisions. For Δb_j with $j \in J$, Equation 5.52 requires that $h_{ij} \Delta b_j \geq -X_{Bi}$ where $i = 1, 2, \ldots, m$ and h_{ij} is the ith row and jth column of the matrix H. The above inequality can be fulfilled by

$$L_j \leq \Delta b_j \leq R_j. \tag{5.53}$$

The bounds are to be determined as follows:

$$\begin{aligned} L_j &= \max[-X_{Bi}/h_{ij};\ h_{ij} > 0] \\ &= -\infty, \quad \text{if all } h_{ij} \leq 0 \end{aligned} \tag{5.54}$$

$$\begin{aligned} R_j &= \min[-X_{Bi}/h_{ij};\ h_{ij} < 0] \\ &= \infty, \quad \text{if all } h_{ij} \geq 0, \end{aligned} \tag{5.55}$$

where the maximization and minimization are taken over $i = 1, 2, \ldots, m$. (Note that $L_j \leq 0$ and $R_j \geq 0$ are always true as evidenced by $X_B \geq 0$.) Chosen according to Equation 5.53, Δb_j is feasible for the constraint Equation 5.41. To satisfy Equation 5.42 and make the smallest $y_i b_j < 0$, one must select Δb_j as follows:

$$\Delta b_j = 0, \quad \text{if } y_j = 0 \tag{5.56}$$

$$= \max[f_j, L_j], \quad \text{if } y_j > 0 \tag{5.57}$$

$$= \min[g_j, R_j], \quad \text{if } y_j < 0. \tag{5.58}$$

Note that the maximal selection is always negative but the minimal one is always positive.

The steady state may be replaced by a simpler expression $\Delta P = 0$. They are equivalent because

$$\Delta P = \sum_j y_j \Delta b_j = -\sum |y_j \Delta b_j| = 0, \tag{5.59}$$

which implies that

$$y_j \Delta b_j = 0, \quad \text{for all } j \in J. \tag{5.60}$$

This is true only when $\Delta b_j = 0$ for all $j \in J$ due to the selection rule that $\Delta b_j = 0$ when $y_j = 0$.

5.7.6.2 *Implementation Algorithm*

The algorithm for sensitivity analysis with changes in the parameter b_i can be implemented by using the following steps:

Step 1. Obtain B, X_B, C, and b from the LP problem, Equation 5.10.

Step 2. Calculate $P = C_B^T C_B$, $H = B^{-1} = [h_{ij}]$, and then $y = H^T C_B$.

Step 3. Identify J and $[f_j, g_j]$ from Equation 5.13 of the problem.

Step 4. Set $\Delta P = 0$.

Step 5. Do the following Steps 5.1 through 5.3. For $j \in J$ and $i = 1, 2, \ldots, m$

5.1 If $y_j = 0$, then set $\Delta b_j = 0$.

5.2 If $y_j > 0$, then set

$$\begin{aligned} L_j &= \max [-X_{Bi}/h_{ij};\ h_{ij} > 0] \\ &= -\infty \quad \text{if all } h_{ij} \leq 0 \end{aligned}$$

Then set $\Delta b_j = \max[f_j, L_j]$.

5.3 If $y_j < 0$, then set

$$\begin{aligned} R_j &= \min [-X_{Bi}/h_{ij};\ h_{ij} < 0] \\ &= \infty \quad \text{if all } h_{ij} \geq 0, \end{aligned}$$

Then set $\Delta b_j = \min[g_j, R_j]$.

Step 6. Update

$$\begin{aligned} \Delta P &= \Delta P + y_j \Delta b_j \\ b_j &= b_j + \Delta b_j \\ f_j &= f_j - \Delta b_j \\ g_j &= g_j - \Delta b_j. \end{aligned}$$

Step 7. Update

$$X_{Bi} = X_{Bi} + h_{ij} \Delta b_j, \quad \text{for all } i = 1, 2, \ldots, m.$$

Step 8. Update

$$P = P + \Delta P.$$

Step 9. Go to step 10 if $\delta P = 0$; go back to step 4 otherwise.

Step 10. Stop with X_B, b, and P as a solution for the postoptimization as formulated.

Finally, it is easy to understand the update performed in steps 6 and 7 except for the intervals. The new intervals must be shifted by the amount of Δb_j to the left if $\Delta b_j > 0$ and to the right if $\Delta b_j < 0$. The update in steps 6 and 7 may be exempted if $\Delta b_j = 0$.

Illustrative example

Minimize $[P = X_1 + X_2 + X_3]$

Subject to

$X_1 + X_3 + X_4 - X_5 = 3 = b_1$

$X_2 - X_3 + X_4 + X_6 = 1 = b_2$

and

$\forall i, \quad X_i \geq 0.$

It is required to further minimize P for $-1 \leq \Delta b_1 \leq 1$ and $0 \leq \Delta b_2 \leq 2$ after a minimized solution has been achieved for the problem. This is a problem of postoptimization and can be solved as follows:

Solution

Step 1. From the primal problem

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad X_B = \begin{bmatrix} X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad C_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Step 2. $P = X_3 = 1$.

$$H = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$y = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Step 3. $J = \{1,2\}$.

$$f_1 = -1, \quad f_2 = 0$$

$$g_1 = 1, \quad g_2 = 2.$$

Step 4. $\Delta P = 0$.

Step 5. With $j=1$, then with
Step 5.2 with $y_1=0.5$, we get

$$L_1 = \max\left[-\frac{1}{0.5}, -\frac{2}{0.5}\right] = -2$$

$$\Delta b_1 = \max[-1, -2] = -1$$

$$\Delta P = 0 + \left(\frac{1}{2}\right)(-1) = -\frac{1}{2}$$

$$b_1 = 3 + (-1) = 2$$

$$f_1 = -1 - (-1) = 0$$

$$g_1 = 1 - (-1) = 2$$

with Step 5.3,

$$X_{B1} = 1 + 0.5(-1) = 0.5$$
$$X_{B2} = 2 + 0.5(-1) = 1.5.$$

Step 6. With $j=2$, then

$$y_2 = -0.5$$

$$R_2 = \min\left[\frac{-0.5}{-0.5}\right] \quad \text{and} \quad \Delta b_2 = \min[2, 1] = 1.$$

Step 7. $\Delta P = -0.5 + (-0.5)\,(1) = -1$

$$b_2 = 1 + 1 = 2, \quad g_2 = 2 - 1 = 1$$
$$f_2 = 0 - 1 = -1.$$

Step 8. $X_{B1} = 0.5 + (-0.5)\,(1) = 0$

$$X_{B2} = 1.5 + (0.5)(1) = 2.$$

Step 9. $P = 1 + (-1) = 0$.

Step 10. $\Delta P = -1 \neq 0$.

10.1 $\Delta P = 0$.

10.2 $j = 1$, $y_1 = \frac{1}{2}$: $\Delta b_1 = \max[0, 0] = 0$, no update.

10.3 $j = 2$, $y_2 = -\frac{1}{2}$: $\Delta b_2 = \min[1, 0] = 0$, no update.

Therefore, $\Delta P = 0$.

Step 11. The optimal solution to the problem is

$$X_B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \text{and} \quad P = 0.$$

Notably, if we change b_2 first and then b_1 or $J = \{2, 1\}$, the solution becomes

$$X_B \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \text{and} \quad P = 0.$$

In fact, the problem has an infinite number of solutions:

$$X_B = \begin{bmatrix} 0 \\ \beta \end{bmatrix}, \quad b = \begin{bmatrix} \beta \\ \beta \end{bmatrix}, \quad \text{and} \quad P = 0,$$

where $2 \leq \beta \leq 3$.

5.7.6.3 Duality in Postoptimal Analysis

The dual of the primal LP problem given by Equations 5.12 through 15.14 can be restated in the form:

$$\max \{D = b^T y \text{ s.t. } A^T y \leq C;\ y \geq 0\}. \tag{5.61}$$

Let the vectors in A and components in C be arranged as

$$A = [B, N] \tag{5.62}$$

and

$$C = \begin{bmatrix} C_B \\ C_N \end{bmatrix}. \tag{5.63}$$

Then, it follows that

$$AX = [B, N] \begin{bmatrix} X_B \\ 0 \end{bmatrix} = BX_B = b \tag{5.64}$$

and

$$P = C^T X = [C_B^T, C_N^T] \begin{bmatrix} X_B \\ 0 \end{bmatrix} = C_B^T X_B. \tag{5.65}$$

Introduce a new m-vector u such that

$$Bu = a_k(A_k), \tag{5.66}$$

where a_k is a vector in N.

The vector y in Equation 5.62 is feasible for the dual problem if

$$N^{\mathrm{T}}y \leq C_{\mathrm{N}}. \tag{5.67}$$

This is true because

$$A^{\mathrm{T}}y = \begin{bmatrix} B^{\mathrm{T}}y \\ N^{\mathrm{T}}y \end{bmatrix} \leq \begin{bmatrix} C_{\mathrm{B}} \\ C_{\mathrm{N}} \end{bmatrix} = C. \tag{5.68}$$

To show Equation 5.69, we assume the contrary to have

$$a_k^{\mathrm{T}}y - C_{\mathrm{N}k} > 0. \tag{5.69}$$

Multiply Equation 5.67 by a positive number θ and then subtract it from Equation 5.65:

$$B(X_{\mathrm{B}} - \theta u) + \theta a_k = b. \tag{5.70}$$

We obtain from Equations 5.66 and 5.67 that

$$y^{\mathrm{T}}Bu = y^{\mathrm{T}}a_k = C_B^{\mathrm{T}}u = a_k^{\mathrm{T}}y$$

and hence

$$a_k^{\mathrm{T}}y - C_{\mathrm{N}K} = C_{\mathrm{B}}^{\mathrm{T}}u - C_{\mathrm{N}K}. \tag{5.71}$$

Multiply Equation 5.71 by θ and then subtract it from Equation 5.66:

$$P - \theta\left(a_k^{\mathrm{T}}y - C_{\mathrm{N}K}\right) = C_{\mathrm{B}}^{\mathrm{T}}(X_{\mathrm{B}} - \theta u) + \theta C_{\mathrm{N}k}. \tag{5.72}$$

Choose

$$\theta = \frac{X_{\mathrm{B}r}}{u_r} = \min\left\{\frac{X_{\mathrm{B}i}}{u_i} : u_i > 0\right\} \tag{5.73}$$

which always exists since $X_{\mathrm{B}} > 0$ (nondegenerate). With θ so chosen, Equation 5.71 demonstrates that a new feasible basis is formed by replacing a_r in B with a_k in N. On the other hand, Equation 5.72 indicates that the new P is decreased from the old value by

$$\theta\left(a_k^{\mathrm{T}}y - C_{\mathrm{N}K}\right) > 0. \tag{5.74}$$

This contradicts the fact that B is a minimal basis.

For any feasible X of Equation 5.10 and its dual y, we have

$$P = C^{\mathrm{T}}X \geq (A^{\mathrm{T}}y)^{\mathrm{T}}X = y^{\mathrm{T}}AX = y^{\mathrm{T}}b = D. \tag{5.75}$$

But, for $X = X_B$, it becomes

$$P = C_B^T X_B = (B^T y)^T X_B = y^T B X_B \tag{5.76}$$

$$= y^T b = D. \tag{5.77}$$

Therefore y is also a maximum of the dual problem.

5.8 Power Systems Applications

Consider a subtransmission system in which certain bus voltages of interest form the vector $|V|$. Assume that the increase in bus voltage magnitude is linearly proportional to injected reactive power at several buses, denoted by the vector ΔQ. The dimensions of vectors $|V|$ and ΔQ may not be the same since the capacitor placement may occur at a different number of buses compared to the buses at which $|V|$ is supported. Then

$$|V| = B\Delta Q, \tag{5.78}$$

where B involves elements from the inverse of the $\partial Q/\partial |V|$ portion of the Jacobian (under the assumptions of superposition and decoupled power flow). To ensure high enough bus voltage,

$$|V| + \Delta|V| \geq |V_{min}|, \tag{5.79}$$

where $|V_{min}|$ is a vector of minimum bus voltage magnitudes and $|V|$ is the "base case" (i.e., no capacitive compensation) bus voltage profile. The $\Delta|V|$ term stems from capacitive compensation. As a result

$$|\Delta V| = B\Delta Q \geq |V_{min}| - |V|. \tag{5.80}$$

Again the concept of a vector inequality in Equation 5.68 is said to hold when each scalar row holds. Furthermore, it is desired to minimize c_q,

$$c_q = c^t \Delta Q. \tag{5.81}$$

c^t is a row vector of 1s commensurate dimension with ΔQ. The cost function, c_q, is a scalar. The minimization of c_q subject to Equation 5.68 is accomplished by LP. In most LP formulations, the inequality constraints are written with the solution vector appearing on the "smaller than" side (i.e., opposite to inequality Equation 5.70), and the index that is extremized is maximized rather than minimized. Both problems are avoided by working with $\Delta Q'$, rather than ΔQ, where

$$\Delta Q' = K - \Delta Q. \tag{5.82}$$

In this discussion, ΔQ entries are assumed to be positive for shunt capacitive compensation.

It is possible also to introduce capacitor costs, which depend on the size of the unit. This is done by allowing other than unity weighting in c^t. Upper limits may also be introduced, but these are usually not needed.

5.9 Illustrative Examples

Example 5.9.1

A subsystem has two generators. There are four key lines with limits given by

$$P_{L1}^{max} = 12 \text{ MW}, \quad P_{L2}^{max} = 12 \text{ MW}$$
$$P_{L3}^{max} = 18 \text{ MW}, \quad P_{L4}^{max} = 18 \text{ MW}.$$

The sensitivity relation between the key lines and the generators is given below:

$$P_{L1} = 2P_{G1} + 3P_{G2}, \quad P_{L2} = 3P_{G1} + 2P_{G2}$$
$$P_{L3} = 6P_{G1} + P_{G2}, \quad P_{L4} = P_{G1} + 6P_{G2}.$$

The system benefit function is $F = P_{G1} + 2P_{G2}$. Find the maximum benefit value of the system using LP.

Solution

$$P_{L1}^{max} = 12, \quad P_{L2}^{max} = 12$$
$$P_{L3}^{max} = 18, \quad P_{L4}^{max} = 18$$
$$P_{L1} = 2P_{G1} + 3P_{G2} \rightarrow 2P_{G1} + 3P_{G2} \leq 12$$
$$P_{L2} = 3P_{G1} + 2P_{G2} \rightarrow 3P_{G1} + 2P_{G2} \leq 12$$
$$P_{L3} = 6P_{G1} + P_{G2} \rightarrow 6P_{G1} + P_{G2} \leq 18$$
$$P_{L4} = P_{G1} + 6P_{G2} \rightarrow P_{G1} + 6P_{G2} \leq 18$$
$$F = P_{G1} + 2P_{G2} \rightarrow \text{maximize}.$$

If we change the variables to be $P_{G1} \rightarrow x_1$, $P_{G2} \rightarrow x_2$ then the problem will be

Maximize $F = x_1 + 2x_2$

Subject to

$2x_1 + 3x_2 \leq 12$

$3x_1 + 2x_2 \leq 12$

$6x_1 + x_2 \leq 18$

$x_1 + 6x_2 \leq 18.$

Solution

Change the inequality constraints to be equality by adding a slack variable to each inequality constraint. Now the problem will be

Maximize $F = x_1 + 2x_2$

Subject to

$2x_1 + 3x_2 + x_3 = 12$

$3x_1 + 2x_2 + x_4 = 12$

$6x_1 + x_2 + x_5 = 18$

$x_1 + 6x_2 + x_6 = 18$

$$C = [1 \quad 2 \quad | \quad 0 \quad 0 \quad 0 \quad 0] = [\bar{C}_B \mid C_B]$$

$$A = \left[\begin{array}{cc|cccc} 2 & 3 & 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 1 & 0 & 0 \\ 6 & 1 & 0 & 0 & 1 & 0 \\ 1 & 6 & 0 & 0 & 0 & 1 \end{array}\right] = [\bar{B} \mid B]$$

$$x = [x_1 \quad x_2 \quad | \quad x_3 \quad x_4 \quad x_5 \quad x_6]^T$$

$$x_B = [x_3 \quad x_4 \quad x_5 \quad x_6]^T = [12 \quad 12 \quad 18 \quad 18]^T$$

$$Z - \bar{C}_B = C_B B^{-1} \bar{B} - \bar{C}_B = [0 \quad 0] - [1 \quad 2] = [-1 \quad -2].$$

First iteration: $k = 2$ implies that the second vector of $\bar{B}$, x_2 is to enter the basis

$$y = B^{-1} \bar{a}_2 = j \begin{bmatrix} 3 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

$$\theta = \min\left\{\frac{x_{Bi}}{y_i}, y_i > 0\right\} = \min\left\{\frac{12}{3}, \frac{12}{2}, \frac{18}{1}, \frac{18}{6}\right\} = 3,$$

$r = 4$, $\theta = 3$; therefore, the fourth vector X_6 is to leave the basis. The new basis after replacement X_2 and X_6 is

$$x_{B\,new} = [x_B - \theta y,\ x_{Br} = \theta] = \{12 - 3 \times 3,\ 12 - 2 \times 3,\ 18 - 3 \times 1,\ 3\}^T$$
$$= [6,\ 6,\ 15,\ 3]^T.$$

New values

$$B = \begin{bmatrix} x_3 & x_4 & x_5 & x_2 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 6 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} x_1 & x_6 \\ 2 & 0 \\ 3 & 0 \\ 6 & 0 \\ 1 & 1 \end{bmatrix}$$

$$C_B = [0 \quad 0 \quad 0 \quad 2]$$
$$x_B = [6,\ 6,\ 15,\ 3]^T$$
$$\bar{C}_B = [1 \quad 0].$$

Components of B^{-1}, $r = 4$, $y_r = 6$,

$$b_{ij} = b_{ij} - \frac{y_i}{y_r} b_{4j}, \quad i \neq 4$$

$$b_{ij} = \frac{y_i}{y_4} b_{4j}, \quad i = 4$$

$$b_{11} = b_{11} - \frac{y_1}{y_4} b_{41} = 1$$

$$b_{12} = b_{12} - \frac{y_1}{y_4} b_{42} = 0$$

$$b_{13} = b_{13} - \frac{y_1}{y_4} b_{43} = 0$$

$$b_{14} = b_{14} - \frac{y_1}{y_4} b_{44} = -\frac{3}{6}(1) = -0.5$$

$$b_{21} = b_{21} - \frac{y_2}{y_4} b_{41} = 0$$

$$b_{22} = b_{22} - \frac{y_2}{y_4} b_{42} = 1$$

$$b_{23} = b_{23} - \frac{y_2}{y_4} b_{43} = 0$$

$$b_{24} = b_{24} - \frac{y_2}{y_4} b_{44} = 0 - \frac{2}{6}(1) = -\frac{1}{3}$$

$$b_{31} = b_{31} - \frac{y_3}{y_4} b_{41} = 0$$

$$b_{32} = b_{32} - \frac{y_3}{y_4} b_{42} = 0$$

$$b_{33} = b_{33} - \frac{y_3}{y_4} b_{43} = 1 - \frac{1}{6}(0) = 1.0$$

$$b_{34} = b_{34} - \frac{y_3}{y_4} b_{44} = 0 - \frac{1}{6}(1) = -\frac{1}{6}$$

$$b_{41} = \frac{b_{41}}{y_4} = 0, \quad b_{42} = 0, \quad b_{43} = 0, \quad b_{44} = \frac{1}{6}.$$

$$B^{-1} = \begin{bmatrix} 1 & 0.0 & 0.0 & -0.5 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{6} \end{bmatrix}.$$

Second iteration

$$Z = C_B B^{-1} \bar{B} = [0 \quad 0 \quad 0 \quad 2] \begin{bmatrix} 1 & 0 & 0.0 & -0.5 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 0 \\ 6 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 0 \\ 6 & 0 \\ 11 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$Z - \bar{C}_B = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \end{bmatrix} - [1 \quad 0] = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

Then x_1 is to enter the basis

$$y = B^{-1} \bar{a} = \begin{bmatrix} 1 & 0 & 0.0 & -0.5 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 6 \\ \frac{1}{6} \end{bmatrix}$$

$$\theta = \min\left\{\frac{x_{Bi}}{y_i}, y_i > 0\right\} = \min\left\{\frac{6}{2} \; \frac{6}{6} \; \frac{15}{6} \; 18\right\} = 1,$$

$r = 2$, $\theta = 1$. Therefore, the second column (x_4) is to leave the basis. x_1 replaces x_4, $y_r = y_2 = 6$.

The new basis is

$$x_{B\,\text{new}} = \{X_B - \theta y, x_{B_r} = \theta\} = \left\{6 - 2, 1, 15 - 6, 3 - \frac{1}{6}\right\} = \left\{4, 1, 9, \frac{17}{6}\right\}.$$

B^{-1} components are

$$b_{ij} = b_{ij} - \frac{y_i}{y_r} b_{rj}, \quad i \neq r, \quad b_{ij} = b_{ij} - \frac{y_i}{y_2} b_{2j},$$

$$b_{ij} = \frac{y_{rj}}{y_r}, \quad b_{ij} = \frac{b_{2j}}{y_2},$$

$$b_{11} = b_{11} - \frac{y_1}{y_2} b_{21} = 1 - \frac{2}{6}(0) = 1$$

$$b_{12} = b_{12} - \frac{y_1}{y_2} b_{22} = 0 - \frac{2}{6}(1) = -\frac{1}{3}$$

$$b_{13} = b_{13} - \frac{y_1}{y_2} b_{23} = 0$$

$$b_{14} = b_{14} - \frac{y_1}{y_2} b_{24} = 3.0 - \frac{2}{6}(2) = \frac{7}{3}$$

$$b_{21} = \frac{b_{21}}{y_2} = 0, \quad b_{22} = \frac{b_{22}}{y_2} = \frac{1}{6}, \quad b_{23} = 0$$

$$b_{24} = \frac{b_{24}}{y_2} = \frac{2}{6} = \frac{1}{3}$$

$$b_{31} = b_{31} - \frac{y_3}{y_2} b_{21} = 0$$

$$b_{32} = b_{32} - \frac{y_3}{y_2} b_{22} = 0 - \frac{6}{6}(1) = -1$$

$$b_{33} = b_{33} - \frac{y_3}{y_2} b_{23} = 1 - \frac{6}{6}(0) = 1.0$$

$$b_{34} = b_{34} - \frac{y_3}{y_2} b_{24} = 1 - (1)2 = -1$$

$$b_{41} = b_{41} - \frac{y_4}{y_2} b_{21} = 0 - \frac{1/6}{6}(0) = 0$$

$$b_{42} = b_{42} - \frac{y_4}{y_2} b_{22} - \frac{1/6}{6}(1) = -\frac{1}{36}$$

$$b_{43} = b_{43} - \frac{y_4}{y_2} b_{23} = 0, \quad b_{44} = b_{44} - \frac{y_4}{y_2} b_{24} = 6 - \frac{1}{36}(2) = \frac{107}{18}$$

$$B^{-1} = \begin{array}{c} \begin{matrix} x_3 & x_1 & x_5 & x_2 \end{matrix} \\ \begin{bmatrix} 1 & -\frac{1}{3} & 0 & \frac{7}{3} \\ 0 & \frac{1}{6} & 0 & \frac{1}{3} \\ 0 & -1 & 1 & -1 \\ 0 & -\frac{1}{36} & 0 & \frac{107}{18} \end{bmatrix} \end{array} \begin{matrix} \\ \\ \\ x_4 \ x_6 \end{matrix}$$

$$B = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 3 & 0 & 2 \\ 0 & 6 & 1 & 6 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x_b = \left[4, 1, 9 \ \frac{17}{6}\right]^T$$

$$C_B = [0 \quad 1 \quad 0 \quad 2]$$

$$\bar{C}_B [0 \quad 0].$$

Third iteration

$$Z = C_B \bar{B}^1 \bar{B} = [0 \quad 1 \quad 0 \quad 2] \begin{bmatrix} 1 & -\frac{1}{3} & 0 & \frac{7}{3} \\ 0 & \frac{1}{6} & 0 & \frac{1}{3} \\ 0 & -1 & 1 & -1 \\ 0 & -\frac{1}{36} & 0 & \frac{107}{18} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{2}{28} & 0 & \frac{110}{9} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{9} & \frac{110}{9} \end{bmatrix}$$

$$Z - \bar{C}_B = \begin{bmatrix} \frac{1}{9} & \frac{110}{9} \end{bmatrix} - [0 \quad 0] = \begin{bmatrix} \frac{1}{9} & \frac{110}{9} \end{bmatrix} > 0,$$

which means that we reached the optimum point,

$$\min P = C_B x_B = [0 \quad 1 \quad 0 \quad 2] \begin{bmatrix} 4 \\ 1 \\ 9 \\ 17/6 \end{bmatrix} = \frac{20}{3},$$

$$x_1 = 1, \quad x_2 = (17/6).$$

Then as a result, $P_{G1} = 1$ and $P_{G2} = (17/6)$.

Example 5.9.2

(a) Use LP to minimize $F = 4x_1 + 5x_2$ subject to

$$x_1 + x_2 \leq 6$$
$$x_1 + 3x_2 \leq 15$$
$$2x_1 + x_2 \leq 12.$$

(b) Determine the solution of if the RHS vector is changed to $[10 \quad 21 \quad 16]^T$.

Solution to Part (a)

Changing inequality constraints to be equality by adding slack:

$$\text{Min } F = 4x_1 + 5x_2$$
$$\text{s.t. } x_1 + x_2 + x_3 = 6$$
$$x_1 + 3x_2 + x_4 = 15$$
$$2x_1 + x_2 + x_5 = 12$$

$$C = [4 \quad 5 \quad | \quad 0 \quad 0 \quad 0] = [\bar{C}_B \mid C_B]$$

$$A = \left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{array}\right] = [\bar{B} \mid B]$$

$$x = [x_1 \quad x_2 \mid x_3 \quad x_4 \quad x_5]\, x_B = [x_3 \quad x_4 \quad x_5]^T = [6 \quad 15 \quad 12]^T$$

$$Z - \bar{C}_B = C_B B^{-1} \bar{B} - \bar{C}_B = [-4 \quad -5].$$

First iteration

Now $k = 2$ implies the second vector of $\bar{B}$ and x_2 enters bases

$$y = B^{-1} \bar{a}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\theta = \min\left\{\frac{x_{Bi}}{y_i}, y_i > 0\right\} = \min\left\{\frac{6}{1}, \frac{15}{3}, \frac{12}{1}\right\} = [1 \quad 5 \quad 7]^T.$$

New values

$$B = \begin{bmatrix} x_2 & x_1 & x_4 \\ 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} x_1 & x_4 \\ 1 & 0 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

$$C_B = [0 \quad 5 \quad 0] \quad \bar{C}_B = [4 \quad 0]$$

$$x_B = [1 \quad 5 \quad 7]^T.$$

Components of B^{-1} were calculated from the matrix inversion lemma to get:

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Second iteration

$$Z = C_B B^{-1} \bar{B} = [0 \quad 5 \quad 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 0 \end{bmatrix} = [5 \quad 5].$$

Since $Z - \bar{C}_B = [1 \quad 5] > 0$, we have reached optimum point at

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

and the corresponding objective value is

$$\text{Min } F = C_B x_B = [0 \quad 5 \quad 0] \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} = 25.$$

Solution to Part (b)

Using perturbation for $b = \begin{bmatrix} 10 \\ 21 \\ 16 \end{bmatrix}$

given $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $x_B = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$, $C_B = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$

$$P = C_B^T x_B = [0 \quad 5 \quad 0] \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} = 25$$

$$H = B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ (from Part (a))}$$

$$y = H^T C_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$$J = \{1, 2, 3\}$$

$$\Delta b = b_2 - b_1 = \begin{bmatrix} 10 \\ 21 \\ 16 \end{bmatrix} - \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 9 \\ 16 \\ 9 \end{bmatrix}$$

$$f_1 \le 9 \le g_1; \quad f_1 = 5; \quad g_1 = 10$$
$$f_2 \le 16 \le g_2; \quad f_2 = 0; \quad g_2 = 10;$$
$$f_3 \le 9 \le g_3; \quad f_3 = 1; \quad g_3 = 10$$

$$\Delta P = 0$$

with $j = 1$, then $y_1 = 0$, $\Delta b_1 = 0$
with $j = 2$, then $y_2 = 5$, we get

$$L_2 = \left[\frac{-x_{Bi}}{h_{i2}}, h_{i2} > 0\right] = \left[\frac{-5}{1}\right] = -5$$

$$\Delta b_2 = -5$$

$$\Delta P = \Delta P + y_j \Delta b_j = 0 + y_2 \Delta b_2 = -25$$

$$b_2 = b_2 + \Delta b_2 = 21 + (-5) = 16$$

With $f_j = f_j - \Delta b_j$, $f_2 = 0 - (-5) = 5$

With $g_j = g_j - \Delta b_j$, $f_2 = 0 - (-5) = 5$

$$x_{B1} = X_{B1} + h_{12} \Delta b_2 = 1 + 0(-5) = 1$$

$$x_{B2} = x_{B2} + h_{22} \Delta b_2 = 5 + 1(-5) = 0$$

$$x_{B3} = x_{B3} + h_{32} \Delta b_2 = 7 + 0(-5) = 7$$

With $j = 3$, $y_3 = 0$ therefore $\Delta b_3 = 0$, then

updating $P = P + \Delta P = 25 + (-25) = 0$.

Second iteration

$$x_B = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 16 \\ 16 \end{bmatrix},$$

$$f_1 = 5, \quad g_1 = 10$$

$$f_2 = 5, \quad g_2 = 21$$

$$f_3 = 1, \quad g_3 = 10$$

$$P = C_B x_B = [0 \quad 5 \quad 0] \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} = 0$$

$$C_B = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

with $j = 1$, then $y_1 = 0$ therefore $\Delta b_1 = 0$
with $j = 2$, then $y_2 = 5$, we get

$$L_2 = \text{Max}\left[\frac{-x_{Bi}}{h_{i2}}, h_{i2} > 0\right] = \text{Max}\left[\frac{-0}{1}\right] = 0$$

Therefore $\Delta b_2 = 0$.

With $j = 3$, $y_3 = 0$ therefore $\Delta b_3 = 0$. And since $\Delta b_1 = \Delta b_2 = \Delta b_3 = 0$, no further update is required.

Therefore $\Delta P = 0$ and the optimal solution to the problem is $x_B = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}$ and $P = 0$.

5.10 Conclusion

This chapter covered LP, one of the most famous optimization techniques for linear objectives and linear constraints. In Section 5.1, the formulation of the natural model associated with the basic assumptions was presented and a graphical solution of the LP problem demonstrated. In Section 5.2 the simplex algorithm was presented and supported with illustrative problems together with a summary of the computational steps involved in the algorithms. The matrix approach solution to the LP problem was presented in Section 5.3 where the formulation of the problem and the revised simplex algorithm were also presented. Duality in LP was presented in Section 5.4.

For those cases where the variables take either integer or continuous values, MIP was presented in Section 5.5 as a way of solving such problems. The branch-and-bound technique for solving this problem was explained. In Section 5.6, the sensitivity method for analysis of postoptimization of LP was presented supported with a method of solution and detailed algorithms. In Section 5.7, a power system application was presented where improvement of a voltage profile using reactive power resources installed in the system was shown. The construction of the method and the solution technique were explained.

5.11 Problem Set

PROBLEM 5.11.1

Given the objective function

$$f(x) = x_1 + 2x_2, \quad x_1, x_2 \geq 0$$

subject to the constraints:

$$\phi_1(x) = 2x_1 + 3x_2 - 12 \leq 0$$
$$\phi_2(x) = 3x_1 + 2x_2 - 12 \leq 0$$
$$\phi_3(x) = 6x_1 + x_2 - 18 \leq 0$$
$$\phi_4(x) = x_1 + 6x_2 - 18 \leq 0,$$

find the point (x) by obtaining the candidate points by finding all possible solutions to be the boundary equations implied by the constraints and testing to satisfy the domain of feasibility.

PROBLEM 5.11.2

Given the objective function

$$f(x) = 4.25x_1 + 4.00x_2, \quad x_1, x_2 \geq 0$$

subject to the constraints:

$$5.00x_1 + 3.00x_2 \leq 15.00$$
$$3.50x_1 + 5.00x_2 \leq 17.50$$
$$4.00x_1 + 3.50x_2 \leq 14.00$$

find the point (x) that maximizes f. Determine (x) to two decimal places.

PROBLEM 5.11.3

Solve the following LP.

Maximize $z = 4x_1 + 6x_2 + 2x_3$

Subject to

$4x_1 - 4x_2 \leq 5$

$-x_1 + 6x_2 \leq 5$

$-x_1 - 3x_2 + 2x_3 \leq 3.$

x_1, x_2, and x_3 are nonnegative integers.

Compare the rounded optimal solution and the integer optimal solution.

PROBLEM 5.11.4

Convert the following problem to standard form and solve.

Maximize $x_1 + 4x_2 + x_3$

Subject to

$2x_1 - 2x_2 + x_3 = 4$

$x_1 - x_3 = 1$

$x_2 \geq 0, \quad x_3 \geq 3.$

PROBLEM 5.11.5

Consider the problem:

Maximize $z = x_1 + x_2$

Subject to

$2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30,$

where x_1 and x_2 are nonnegative integers.

Find the optimal noninteger solution graphically. By using IP, show graphically the successive parallel changes in the value that will lead to the optimal integer solution.

PROBLEM 5.11.6

Consider a power system with two generators with cost functions given by

$$\text{Generator 1: } F_1(P_1) = 80 + 7.2P_1 + 0.00107P_1^2 \text{ (\$)}$$
$$\text{Generator 2: } F_2(P_2) = 119 + 7.2P_2 + 0.00072P_2^2 \text{ (\$)}.$$

Where generators 1 and 2 are limited to producing 400 and 600 MW of power, respectively. Given that the system load is 500 MW, then

1. Formulate the problem into LP form.
2. Calculate the optimal generation.
3. Determine the optimal generation cost.

PROBLEM 5.11.7

Consider the problem

Maximize $z = x_1 + x_2$

Subject to

$2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30,$

where x_1 and x_2 are nonnegative integers.

PROBLEM 5.11.8

Consider the following problem:

Maximize $Z = -5x_1 + 5x_2 + 13x_3$

Subject to

$-x_1 + x_2 + 3x_3 \leq 20$

$12x_1 + 4x_2 + 10x_3 \leq 90$

$x_i \geq 0, \quad i = 1, 2, 3.$

Conduct sensitivity analysis by investigating each of the following changes in the original model. Test the solution for feasibility and optimality.

1. Change in the right-hand constraint 1 to

$$b_1 = 30.$$

2. Change in the right-hand constraint to

$$b_2 = 70.$$

3. Change in the right-hand sides to

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 100 \end{bmatrix}.$$

4. Change in the coefficient of x_3 in the objective function to

$$c_3 = 8.$$

5. Change in the coefficient of x_1 to

$$\begin{bmatrix} c_1 \\ a_{11} \\ a_{12} \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}.$$

6. Change in the coefficient of x_2 to

$$\begin{bmatrix} c_2 \\ a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}.$$

7. Introduce a new variable x_6 with coefficients

$$\begin{bmatrix} c_6 \\ a_{16} \\ a_{26} \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 5 \end{bmatrix}.$$

8. Introduce a new constraint $2x_1 + 3x_2 + 5x_3 \leq 50$ (denote the slack variables by x_6).
9. Change constraint 2 to $10x_1 + 5x_2 + 10x_3 \leq 100$.

PROBLEM 5.11.9

Consider the following problem:

Maximize $Z = 2x_1 + 7x_2 - 3x_3$

Subject to

$x_1 + 3x_2 + 4x_3 \leq 30$

$x_1 + 4x_2 - x_3 \leq 10$

$x_i \geq 0, \quad i = 1, 2, 3.$

Reformulate the problem using x_4 and x_5 as slack variables. Conduct sensitivity analysis by investigating each of the following changes in the original model. Test the solution for feasibility and optimality.

1. Change in the right-hand sides to

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \end{bmatrix}.$$

2. Change in the coefficient of x_3 to

$$\begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}.$$

3. Change in the coefficient of x_1 to

$$\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

4. Introduce a new variable x_6 with coefficients

$$\begin{bmatrix} c_6 \\ a_{16} \\ a_{26} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

5. Change in the objective function

$$Z = x_1 + 5x_2 - 2x_3.$$

6. Introduce a new constraint

$$3x_1 + 2x_2 + 3x_3 \leq 25.$$

7. Change constraint 2 to

$$x_1 + 2x_2 + 2x_3 \leq 135.$$

PROBLEM 5.11.10

A manufacturer develops two kinds of products by using three machines types. Each machine has a limited amount of time as shown below:

	Production Time (h/Unit)		
Machines List	x_1	x_2	**Total Time (h)**
Type I	1	1	6
Type II	1	3	15
Type III	2	1	

1. Solve the problem by using LP method with the objective to minimize total production time.
2. Resolve if the total time is perturbed such that $b = \begin{bmatrix} 7 \\ 16 \\ 12 \end{bmatrix}$.

PROBLEM 5.11.11

Maximize $f(x) = -2x_1 - x_2 - x_3$

Subject to $3x_1 + 5x_2 + 2x_3 \geq 6$

$4x_1 - 2x_2 + x_3 \geq 3.$

Do the LP solution by changing the right-hand side to $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$.

PROBLEM 5.11.12

A problem is to maximize $Z = x_1 + x_2$ subject to the constraints given by $2x_1 + 5x_2 \geq 16$ and $6x_1 + 5x_2 \geq 30$ where $x_i \geq 0$ for $i = 1, 2$.

1. Solve for the decision variables by using the simplex method.
2. If the right-hand side is changed by 10%, what will be the new optimal value of the objective function, $\mathbf{Z}^*$ using postoptimal technique?

PROBLEM 5.11.13

Given the primal LP problem to maximize $z = c^T x$ subject to $Ax = b$ with $x \geq 0$, write the complete dual problem formulation. What is the relationship between the values of the objective function in the primal and dual problems?

PROBLEM 5.11.14

Solve the following problem by the revised simplex method:

Maximize $f(x) = 3x_1 - x_2 + 1.5x_3$

Subject to

$2x_1 - x_2 + 2x_3 \leq 2.2$

$x_1 + 4x_3 \leq 4.5$

$x_1, x_2, x_3 \geq 0.$

PROBLEM 5.11.15

Apply the revised simplex method to

Maximize $f(x) = 4x_1 + 2x_2 + 4x_3$

Subject to

$4x_1 + 3x_2 + 8x_3 \leq 10$

$4x_1 + x_2 + 12x_3 \leq 7$

$4x_1 - x_2 + 3x_3 \leq 6$

$x_1, x_2, x_3 \geq 0.$

PROBLEM 5.11.16

Maximize $2x_1 + x_2 + 3x_3 + 4x_4$

s.t. $x_1 + 3x_2 - x_3 + 2x_4 = 6$

$-x_1 + 2x_2 + x_3 + x_4 = 4$

$x_1, x_2, x_3, x_4 \geq 0$

x_1 and x_2 are integers.

PROBLEM 5.11.17

Maximize $z = 2x_1 + 3x_2$

Subject to

$5x_1 + 7x_2 \leq 35$

$4x_1 + 9x_2 \leq 36$

$x_1, x_2 \geq 0$ and integers.

PROBLEM 5.11.18

Maximize $z = 4x_1 + 6x_2 + 2x_3$

Subject to

$4x_1 - 4x_2 \leq 5$

$-x_1 + 6x_2 \leq 5$

$-x_1 + x_2 + x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$ and integers.

Compare the rounded optimal solution and the integer optimal solution.

PROBLEM 5.11.19

Develop and verify the graphical solution to the problem below and determine the existence of an integer solution when the decision variables are discrete:

Maximize $z = 2x_1 + x_2$

Subject to

$10x_1 + 10x_2 \leq 9$

$10x_1 + 5x_2 \geq 1$

$x_1, x_2 \geq 0.$

PROBLEM 5.11.20

Consider the problem

Maximize $z = x_1 + x_2$

Subject to

$2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

$x_1, x_2 \geq 0$ and integers.

Find the optimal noninteger solution graphically. By using the branch-and-bound algorithm, show graphically the successive parallel changes in the objective value that will lead to the optimal integer solution.

References

1. Bialy, H., An elementary method for treating the case of degeneracy in linear programming, *Uternehmensforschung* (Germany), 10(2), 116, 118–123.
2. Bitran, G. R. and Novaes, A. G., Linear programming with a fractional objective function, *Operations Research*, 21(1), 22–29, Jan.–Feb. 1973.
3. Boulding, K. E. and Spivey, W. A., *Linear Programming and the Theory of the Firm*, Macmillan, New York, 1960.
4. Glover, F., A new foundation for a simplified primal integral programming algorithm, *Operations Research*, 16(4), 727–740, July–August 1968.
5. Harris, M. Y., A mutual primal–dual linear programming algorithm, *Naval Research Logistics Quarterly*, 17(2), 199–206, June 1970.
6. Heady, E. O. and Candler, W., *Linear Programming Methods*, Iowa State College Press, Ames, IA, 1958.
7. Hillier, F. S. and Lieberman, G. J., *Introduction to Operations Research*, 4th edn., Holden-Day, San Francisco, CA, 1986.
8. Lavallee, R. S., The application of linear programming to the problem of scheduling traffic signals, *Operations Research*, 3(4), 86–100, 1968.
9. Ravi, N. and Wendell, R. E., The tolerance approach to sensitivity analysis of matrix coefficients in linear programming-I, Working Paper 562, Graduate School of Business, University of Pittsburgh, October 1984.
10. Chieh, H. T., *Applied Optimization Theory and Optimal Control*, Feng Chia University, China, 1990.
11. Luenberger, D. G., *Introduction to Linear and Nonlinear Programming*, Addison-Wesley, Reading, MA, 1984.
12. Hamdy, A., *Taha, Operations Research: An Introduction*, Prentice Hall, Englewood Cliffs, NJ, 8th edn., April 2006.
13. Dantzig, G. B., *Linear Programming and Extension*, Princeton University Press, 1963.
14. Robert Vanderbei, *Linear Programming: Foundations and Extensions*, Springer, 2nd edn., May 2001.

6

Interior Point Methods

6.1 Introduction

Many engineering problems, including the operation of power systems, are concerned with the efficient use of limited resources to meet a specified objective. If these problems can be modeled, they can be converted to an optimization problem of a known objective function subject to given constraints. Most practical systems are nonlinear in nature; however, some approximations are usually tolerable to certain classes of problems. Two methods commonly used are linear and quadratic programming. The former solves those problems where both the objective and constraints are linear in the decision variables [3,4,25]. The quadratic optimization method assumes a quadratic objective and linear constraints.

The well-known simplex method has been used to solve linear programming (LP) problems. In general, it requires burdensome calculations, which hamper the speed of convergence. In an attempt to improve the convergence properties, recent work by Karmarkar [1,2,8,9] on variations of the interior point (IP) method was proposed. The variants include protective, affine-scaling, and path-following methods. Each of these methods solves the LP problem by determining the optimal solution from within the feasible interior (FI) region of the solution space [1,2].

Projective methods are known to require $O(nL)$ iterations. They rely on the projective algorithm, which requires a good scheme. Several schemes have been proposed in the literature. These methods are different from the simplex method, which seeks the optimum solution from a corner point of the solution space.

Affine-scaling methods have no known polynomial time complexity, and can require an exponential number of iterations if they are started close to the boundary of the feasible region. Also, it has been shown that these methods can make it difficult to recover dual solutions and prove optimality when there is degeneracy. However, these methods work well in practice. Very recently, a polynomial time bound for a primal–dual affine method has been obtained.

Path-following methods generally require $O(n0.5L)$ iterations in the worst case, and work by using Newton's method to follow the central path of

optimal solutions obtained by a family of problems defined by a logarithmic barrier function. The most popularly used scheme so far is the barrier method developed by Meggido [13], Kojima [23], and Montiero and Adler [24]. They have shown that the algorithm requires $O(n^3L)$ overall time and no IP algorithm has been shown to have a better worst-case complexity bound. McShane et al. [12] have given a detailed implementation of the algorithm. Their results have been recently adapted to power system problems.

Howard University research contract EPRI—RP2436 extends the results of earlier works [14,16] by improving on the starting and terminating conditions of the IP method for solving LP problems and the extension of the algorithm for solving quadratic-type problems.

The optimal power flow (OPF) problem [5,6] has been recently reviewed [7] as a process of determining the state of power systems that guarantee affordability, reliability, security, and dependability. These abilities optimize given objectives that satisfy a set of physical and operating constraints. In general, these objectives are designated as transmission losses, fuel cost, reactive sources allocation, and voltage feasibility. In general, OPF is a large-scale non-LP problem with thousands of input variables and nonlinear constraints. The problem can be formulated as

Minimize $f(z)$

Subject to

$h(z) = 0$, with $1 \leq z \leq u$.

f and h are continuously differentiate functions in R^n with values in R and R^m, and l and u are vectors in R^n corresponding to lower and upper bounds in the variables, respectively.

The current interest in IP algorithm [12-22] was spanned by searching algorithm for LP which is based on two key ideas:

1. Steepest descent direction is reached more effectively in improving iterate. This is at the center of the polytope forward by the linear constraints than if it were at the boundary.
2. Transformation of the decision space can be found such that it places the iterate at the center of the polytope without altering the problem. Following the work of Karmarkar, several research works have achieved the IP to many variants of IP involving the primal–dual (Mehrotra method). The most popular variant of IP is equivalent to the well-known logarithmic barrier methods which have been discussed in nonlinear programming (NLP). The main idea in all these barrier methods is discussed in this chapter. IP point methods have been proven successful for solving power system OPF problem. It is easily applicable to OPF based on LP technique described in Chapter 5 and is used for quadratic programming and nonlinear-based method derived from the banner method.

6.2 Karmarkar's Algorithm

The new projective-scaling algorithm for LP developed by Karmarkar has caused quite a stir in the optimization community partly because the speed advantage gained by this new method (for large problems) is reported to be as much as 50:1 when compared to the simplex method [8]. This method has a polynomial bound on worst-case running time that is better than ellipsoid algorithms.

Karmarkar's algorithm is significantly different from George Dantzig's simplex method [25] that solves an LP problem starting with one extreme point along the boundary of the feasible region and skips to a better neighboring extreme point along the boundary, finally stopping at an optimal extreme point. Karmarkar's IP rarely visits many extreme points before an optimal point is found. The IP method stays in the interior of the polytope and tries to position a current solution as the "center of the universe" in finding a better direction for the next move. By properly choosing the step lengths, an optimal solution is achieved after a number of iterations. Although this IP approach requires more computational time in finding a moving direction than the traditional simplex method, a better moving direction is achieved resulting in fewer iterations. Therefore, the IP approach has become a major rival of the simplex method and is attracting attention in the optimization community.

Figure 6.1 illustrates how the two methods approach an optimal solution. In this small problem, the projective-scaling algorithm requires approximately the same amount of iterations as the simplex method. However, for a large problem, this method only requires a fraction of the number of iterations that the simplex method would require.

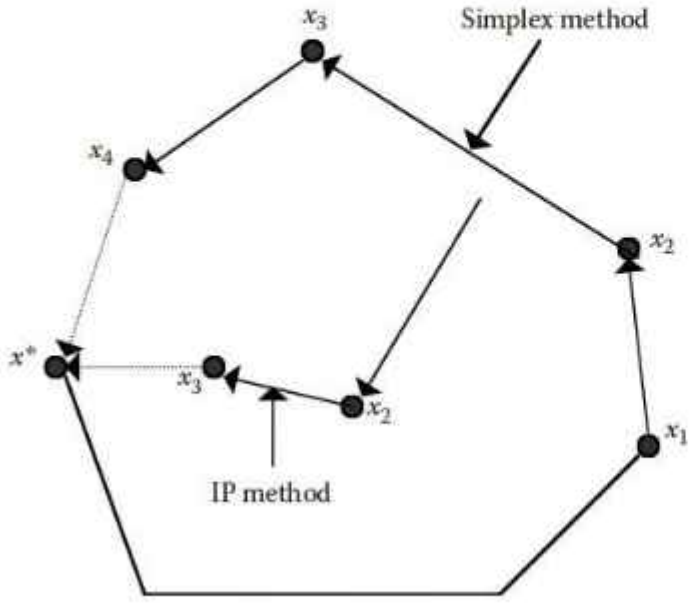


FIGURE 6.1
Illustration of IP and simplex methods.

A major theoretical attraction of the projective-scaling method is its superior worst-case running time (or worst-case complexity). Assume that the size of a problem is defined as the number of bits N required to represent the problem in a computer. If an algorithm's running time on a computer is never greater than some fixed power of N, no matter what problem is solved, the algorithm is said to have polynomial worst-case running time. The new projective-scaling method is such an algorithm.

Due to the results of [10,15], several variants of IPs have been proposed such as the affine-scaling method which is discussed in this chapter. Affine-scaling methods have no known polynomial time complexity, and can require an exponential number of iterations if they are started close to the boundary of the feasible region. It has also been shown that these methods can make it difficult to recover dual solutions and prove optimality when there is degeneracy. However, these methods do work well in practice. Very recently, a polynomial time bound for a primal–dual affine method has been obtained [11].

Path-following methods generally require $O(n0.5L)$ iterations in the worst case, and work by using Newton's method to follow the "central path" of optimal solutions obtained by a family of problems defined by a logarithmic barrier function. Two parameters, the barrier parameter and an underestimated optimal value of the objective function, are the linking parameters between all methods. Barrier methods have been used to construct primal-path-following algorithms, and the method of centers used as a basis for dual algorithms. After scaling has been used to construct both primal and dual algorithms, other variants of barrier methods have been used to construct primal–dual path-following algorithms and an affine variant of the primal–dual algorithms. In general, the above methods, whether projective affine method of centers, or path following, are all simple variants of the algorithm barrier methods applied to the primal, dual, or primal and dual problems together [15].

As mentioned above, since Karmarkar's discovery of the IP method [26,27] and its reported speed advantage obtained over other traditionally used methods, many variants of the IP method have evolved in an attempt to solve the above posed problems. Of these the projective scaling, the dual and primal affine methods, and the barrier function method are the most popular.

These variants of the IP method are presented and evaluated based on their algorithms and the problems they solve.

6.3 Projective-Scaling Method

The projective-scaling algorithm has attracted a great deal of interest due to Karmarkar's ingenious proof that its running time is a polynomial function of the problem size even in the worst case. Karmarkar showed that if n is the

number of variables in LP Problems and L is the number of bits used to represent numbers in the computer, the theoretical worst-case running time is $O(n^{3.5}L^2)$. That is, as the problem size increases, the running time tends to be a constant multiple of $n^{3.5}L^2$, which is substantially better than the ellipsoid algorithm's worst-case running time of $O(n^6L^2)$.

The problems solved by the projective algorithm are in the following form:

Minimize $c^{\mathrm{T}}y$

Subject to

$Ay = 0$

$e_y^{\mathrm{T}} = 1, \quad y \geq 0,$

where

A is an m by n matrix

e is a vector of n ones

The main algorithm for the projective method is presented below.

Algorithm for the Projective-Scaling Method

Step 1. *Initialization.*
$k = 0$, $x^0 = c/n$, and let L be a large positive integer.

Step 2. *Optimality check.*
IF $c^{\mathrm{T}}x^k$ is $\leq 2^{-L}c^{\mathrm{T}}e/n$, **THEN** stop with an optimal solution $x^* = x^k$.
Otherwise go to step 3.

Step 3. *Iterate for a better solution.*
Let $X^{\mathrm{T}} = \mathrm{Diag}(x^k)$

$B^k = \left\{\frac{AX^k}{e^k}\right\}$, e is a matrix.

$d^k = -[I - B_k^{\mathrm{T}}(B_kB_k^{\mathrm{T}})^{-1}B_k]X_kc,$

D is the direction of the real line.

$y^{k+1} = \frac{e}{n} + \frac{\alpha}{n}\left(\frac{d^k}{\|d^k\|}\right), 0 \leq \alpha \leq 1$

$x^{k+1} = \frac{(x_k)y^{k+1}}{(e^{\mathrm{T}}x_k)y^{k+1}}.$

Set $k = k + 1$. Go to step 2

where

x^k is an interior feasible solution

x^n is an n-dimensional diagonal matrix

B^k is the constrained matrix in Karmarkar's standard form

d^k is a feasible direction of the projective negative direction as defined above

y^{k+1} is a new interior feasible solution

L is chosen to be the problem size, where $2^{-L} > e$, $e > 0$

This algorithm terminates in $O(nL)$ iterations. A large value of α tends to speed up the iteration.

6.4 Dual Affine Algorithm

Both the primal and dual interior methods of the variety initiated by Karmarkar can be viewed as special cases of the logarithmic barrier method applied to either the primal or the dual problem. The problems solved by the dual and primal affine methods and their algorithms are presented below. Consider the LP problem given by

Maximize $Z = c^T x$,

Subject to

$Ax \leq b$ and x is unrestricted.

By introducing slack variables into the constraints, the inequality constraints are converted such that we obtain a new formulation given by

Maximize $Z = c^T x$,

Subject to

$Ax + s = b$

$s \geq 0$ and x is unrestricted.

Algorithm

Step 1. *Initialization.*
Set counter value $k=0$ and the tolerance $\in$ (a small positive number).
Obtain a starting solution (x^0, s^0) such that $Ax^0 + s^0 = b$ and $s^0 > 0$.
Set the acceleration constant α, where $0 < \alpha < 1$.

Step 2. *Obtaining the translation direction.*
Computer $d_x^k = (A^T W_k^{-2} A)^{-1} c$, where $W_k = \text{diag}(s_k)$.
Compute the direction vector $d_s^k = -A d_x^k$.

Step 3. *Check for unboundness.*
IF $d_s^k = 0$, **THEN** (x^k, s^k) is the dual optimal. Go to step 9.
IF $d_s^k > 0$, **THEN** the problem is unbounded. Go to step 10.
Otherwise, $d_s^k < 0$. Go to step 4.

Step 4. *Compute the primal estimate y^k.*
$y^k = -W_k^{-2} d_s^k$.

Step 5. *Optimality test.*
IF $y^k \geq 0$ **AND** $(b^T y^k - c^T x^k) \leq \in$, **THEN** (x^k, s^k) is the dual optimal solution and is the primal optimal solution. Go to step 9. **Otherwise**, go to step 6.

Step 6. *Compute the step length* β_k.
$$\beta_k = \min_i \left\{ \frac{\alpha s_i^k}{-(d_s^k)_i} \,\middle|\, _{(d_s^k)_i < 0} \right\}, \quad \text{where } 0 < \alpha < 1.$$

Step 7. *Update the dual variables* (x^k, s^k).
$x^{k+1} = x^k + \beta_k d_x^k$
$s^{k+1} = s^k + \beta_k d_s^k$.

Step 8. *Increment the counter and check for termination.*
IF $k \geq k_{max}$, **THEN** flag the user (maximum number of iteration is reached). Go to step 10.
Otherwise, increment the counter
$k = k + 1$ and go to step 2.

Step 9. *Calculate the final objective value and display the results.*
Optimal solution is $x^* = x^k$ and the objective value is $Z^* = c^T x^*$. Display the final solution.

Step 10. Stop.

6.5 Primal Affine Algorithm

Consider the LP problem given by

Maximize $Z = c^T x$

Subject to

$Ax = b$ and $x \geq 0$ (x is unrestricted).

Algorithm

Step 1. *Initialization.*
Set the iteration limit, k_{max}, and initialize the counter value at $k = 0$.
Select tolerance $\in$ (a small positive number).
Set vector $e = [1, 1, \ldots, 1]$.
Obtain a starting solution $x^0 \geq 0$ such that $Ax^2 = b$.
Set the acceleration constant, α where $0 < \alpha < 1$.

Step 2. *Compute the estimate vectors* w^k.
$w^k = (AD_k^2 A^T)^{-1} AD_k^2 c$ where $D_k = \text{diag}(x^k)$.

Step 3. *Compute the reduced cost coefficient vector.*
$r^k = c - A^T w^k$.

Step 4. *Optimality check.*
IF $r^k = 0$ **AND** $e^T D_k r^k \leq \in s$, **THEN**
x^k is the primal optimal. Go to step 10. **Otherwise**, continue.

Step 5. *Compute translation direction* d_y^k.
$d_y^k = -D_k r^k$

Step 6. *Check for feasibility and constant objective.*
IF $d_y^k > 0$, **THEN** "the problem is unbounded." Go to step 11.
IF $d_y^k = 0$, **THEN** "the primal optimal solution is x^k."
Go to step 10. **Otherwise**, continue.

Step 7. *Compute the step length* β_k.
$\beta_k = \min_i \left\{ \frac{\alpha}{-(d_y^k)_i} \Big|_{(d_y^k)_i < 0} \right\}$, where $0 < \alpha < 1$.

Step 8. *Update the Primal Variables* x^k.
$x^{k+1} = x^k + \beta_k D_k d_y^k$.

Step 9. *Increment the counter and check for termination.*
IF $k \geq k_{max}$, **THEN** flag the user (maximum number of iterations is reached). Go to step 11.
Otherwise, increment the counter
$k = k + 1$ and go to step 2.

Step 10. *Calculate the final objective value and display the results.*
Optimal solution is $x^* = x^k$ and the objective value is $Z^* = c^T x^*$.
Print the solution.

Step 11. *Stop.*

6.6 Barrier Algorithm

Barrier-function methods treat inequality constraints by creating a barrier function, which is a combination of the original objective function and a weighted sum of functions with a positive singularity at the boundary. As the weight assigned to the singularities approaches zero, the minimum of the barrier-function approaches the minimum of the original constrained problem. Barrier-function methods require a strictly feasible starting point for each minimization, and generate a sequence of strictly feasible iterates. The barrier transformation for linear programs with upper and lower bounds on the variables is given in the following form.

Minimize $c^T x$
Subject to
$Ax = b, \quad 1 \leq x \leq u,$

where A is an m by n matrix with $m \leq n$.

When applying the barrier-function method to the problem described above, the subproblem to be solved takes the following form.

$$\underset{x \in R^n}{\text{Minimize}}\ F(x) \equiv c^T x - \mu \sum_{j=1}^{n} \ln x_j$$

Subject to

$$Ax = b,$$

where the scalar μ ($\mu > 0$) is known as the barrier parameter and is specified for each subproblem. The equality constraints cannot be treated by a barrier transformation and, thus, are handled directly. The general algorithm is given below.

Algorithm for the Barrier Method (In Brief)
At the start of each iteration, the quantities μ, x, π, η are known, where $\mu > 0$, $x > 0$, $Ax = b$, and $\eta = c - A^T p$. A correction of π is calculated at each stage since a good estimate is available from the previous iteration. The main steps of the algorithm are as follows.

Step 1. Define $D = \text{Diag}(x_j)$ and compute $r = D\eta - \mu e$.
Note that r is a residual from the optimality condition for the barrier subproblem, and hence $||r|| = 0$ if $x = x^*(m)$.

Step 2. Terminate if μ and $||r||$ are sufficiently small.

Step 3. If appropriate, reduce μ and reset r.

Step 4. Solve the least squares problem
$\underset{\delta\pi}{\text{Minimize}} \|r - DA^T\delta\pi\|$.

Step 5. Compute the updated vectors
$\pi \leftarrow \pi + \delta\pi$ and $\eta \leftarrow \eta - A^T\delta\pi$.
Set $r = D\eta - \mu e$ (the updated scaled residual) and $\pi = -(1/\mu)Dr$.

Step 6. Find α_M, the maximum value of α such that $x + \alpha p \geq 0$.

Step 7. Determine the step length $\alpha \in (0, \alpha_m)$ at which the barrier function $F(x + \alpha p)$ is suitably less.

Step 8. Update $x \leftarrow x + \alpha p$.

All iterates satisfy $Ax = b$ and $\mathbf{x} > \mathbf{0}$. The vectors p and η approximate the dual variables π^* and reduced cost η^* of the original linear program.

6.7 Extended IP Method for LP Problems

The LP problem is formulated in the standard formula of the LP IP method form as follows:

$$\text{Maximize } P = C^{\mathrm{T}}x \tag{6.1}$$

Subject to

$$Ax = b$$

$$b_i > 0, \quad i = 1, \ldots, m, \quad x_i > 0, \quad i = 1, \ldots, m. \tag{6.2}$$

The IP method involves a sequence that consists of a feasible IP ($Ax = b, x > 0$) that makes the objective function increase until it reaches its limit. The limit is an optimal solution of the problem. The IP method utilizes all the vectors in A together with the points in the sequence to generate a maximum increase of the objective function. In addition to programming simplicity, it is superior to the simplex method in computation time and convergence for large systems (large $m > n$).

The condition that $b > 0$ as imposed in the simplex method is waived here and the matrix A is only required to be of full rank $m < n$. Also, by using an appropriate conversion of inequality, two-sided constraints can be handled by the proposed IP method.

Finally, to guarantee existence of the feasible IPs, a trivial condition has been imposed in that the problem has no less than two feasible points ($Ax = b, x > 0$), one of which is a bounded solution for the problem.

6.8 FI Sequence

For convenience, the LP problem can be written as follows:

$$\text{Maximize } a^{\mathrm{T}}x \tag{6.3}$$

Subject to

$$Ax = b,$$

$$\text{Such that } x \geq 0. \tag{6.4}$$

Note C^{T} is replaced by a^{T} where a is a column vector to facilitate description of the IP method. The considered FI sequence contains only feasible and IPs of the problem; that is,

$$S = \{x^1, x^2, \ldots, x^k, x^{k+1}\} \tag{6.5}$$

with

$$Ax^k = b \tag{6.6}$$

and

$$x^k > 0, \quad \text{for all } k > 0. \tag{6.7}$$

For a known point x^k, a diagonal matrix is formed by

$$D = \text{diag}[x_1^k, x_2^k, \ldots, x_n^k], \tag{6.8}$$

where

$$x^k = [x_1^k, x_2^k, \ldots, x_n^k]. \tag{6.9}$$

The FI sequence is then generated recursively according to

$$x^{k+1} = x^k + \beta D d_p, \tag{6.10}$$

where
d_p is an n-vector
β is a positive number

They are to be chosen in such a way that x^{k+1} is a feasible IP wherever x^k is. As such, S contains all FI points of x.

The objective functions and constraints between two consecutive points are related by

$$a^T x^{k+1} = a^T x^k + \beta d^T d_p \tag{6.11}$$

and

$$Ax^{k+1} = Ax^k + \beta B d_p, \tag{6.12}$$

where

$$d = Da \tag{6.13}$$

and

$$B = AD. \tag{6.14}$$

D is determined by x^k and it changes from point to point. Let d be orthogonally decomposed into $d = d_p + d_q$, where d_p is the projection on the null space of B and d_q is in the B^T-subspace (spanned by the vectors of B^T). Then, it follows that

$$Bd_p = 0, \quad d_q = B^T v, \quad \text{and} \quad d_p^T d_q = 0, \tag{6.15}$$

where v is an m-vector, the coordinates of d_q. Solving v and then d_q, we have

$$d_q = B^T(BB^T)^{-1}Bd \tag{6.16}$$

and

$$d_p = d - d_q = [\mathbf{I} - B^T(BB^T)^{-1}B]d = Du, \tag{6.17}$$

where

$$U = a - A^T w \tag{6.18}$$

with

$$w = (BB^T)^{-1}Bd. \tag{6.19}$$

To generate the FI sequence S, the LP problem is divided into two cases: trivial and ordinary. The former has the vector a confined in the A^T space while the latter does not. In the trivial case, there exists an m-vector v such that $a = A^T v$ and hence $a^T x = v^T Ax = vv^T b = \text{constant}$. Thus, all the feasible solutions yield the same objective functions and, hence, there is no optimization involved in the problem. Consequently, it remains only to generate S for the ordinary case.

Starting with a known FI point x^1, S is generated recursively according to Equation 6.10 until reaching a point x at which $d_p = 0$. S contains all the points except x, which is referred to as the limit point of S. The problem is assumed to have no less than two feasible points, and between them there exists a bounded solution of the problem. Since feasible points form a convex set, there are an infinite number of FI points existing within the problem.

For the ordinary case with a bounded solution, one may draw the following conclusions: (1) S contains an infinite number of points and (2) S always has a limit point. To show this, let us assume $d_p = 0$ at a finite k; then $d = d_p + d_q = d_q$ reveals that

$$d = Da = B^T(BB^T)^{-1}Bd = DA^T(BB^T)^{-1}Bd = DA^T v$$

from which we have $a = A^T v$ since D is nonsingular. This is a trivial case and, hence, d_p cannot be zero for finite k.

It follows from Equation 6.11 and $d_q^T d_p = 0$ that

$$a^T x^{k+1} = a^T x^k + \beta \|d_p\|^2. \tag{6.20}$$

Summing both sides of Equation 6.20 from $k = 1$ to $k = N$ gives

$$a^T x^{N+1} = a^T x^1 + \sum_{k=1}^{N} \beta \|d_p\|^2. \tag{6.21}$$

To make Equation 6.21 bounded, it is necessary that $d_p = 0$ as $N \to \infty$. Otherwise, the right side becomes positively unbounded and, hence, x^{N+1} (feasible) yields an objective function, which contradicts the assumption of a bounded solution.

Now, it remains to specify β to make x^{k+1} an FI point if x^k is one. We choose for this purpose

$$0 < \beta < \frac{1}{-\gamma'} \tag{6.22}$$

where γ is the smallest component of d_p. It is asserted that $\gamma < 0$ for all points in S. Indeed, if $\gamma \geq 0$ then $d_p \geq 0$ and $x^{k+1} > 0$ for any $\beta > 0$ according to Equation 6.10. The objective function indicated by Equation 6.20 becomes positively unbounded together with β since $d_p \neq 0$ in S. This contradicts the assumption of a bounded solution and, hence, $\gamma < 0$ must be true.

If x^k is an FI point, then the ith component of Equations 6.10 and 6.12 becomes $x_i^{k+1} = x_i^k + \beta x_i^k d_{p_i} = x_i^k(1 + \beta d_{p_i}) \geq x_i^k(1 + \beta\gamma) > x_i^k(1 - 1) = 0$, for all $i = 1, 2, \ldots, n$ and $Ax^{k+1} = Ax^k + 0 = b$. Since x^{k+1} is also an FI point, all the points of S are FI points if x^1 is by induction. Figure 6.2 shows the IP method algorithm.

6.8.1 Optimality Condition

The limit point of the FI sequence is an optimal solution of the LP problem. This can be shown as follows.

First, we demonstrate that $d_p = 0$ at the limit point implies $u \leq 0$. It follows from Equation 6.17 and $d_p = 0$ that $d_{p_i} = x_i u_i = 0$ for all $i = 1, 2, \ldots, m$. We need only consider the case $u_i > 0$ and $x_i = 0$. In such a case, there exists an $\varepsilon > 0$ and integer N such that $u_i^k > \varepsilon$ for $k \geq N$ where u_i^k is the ith component of Equation 6.18 evaluated at x^k. The ith components of Equations 6.10 and 6.17 are

$$x^{k+1} = x_i^k + \beta_i x_i^k d_{p_i} \tag{6.23}$$

and

$$d_{p_i} = x_i^k u_i^k. \tag{6.24}$$

Then, we obtain

$$\frac{x_i^{k+1}}{x_i^k} = 1 + \beta_i x_i^k u_i^k. \tag{6.25}$$

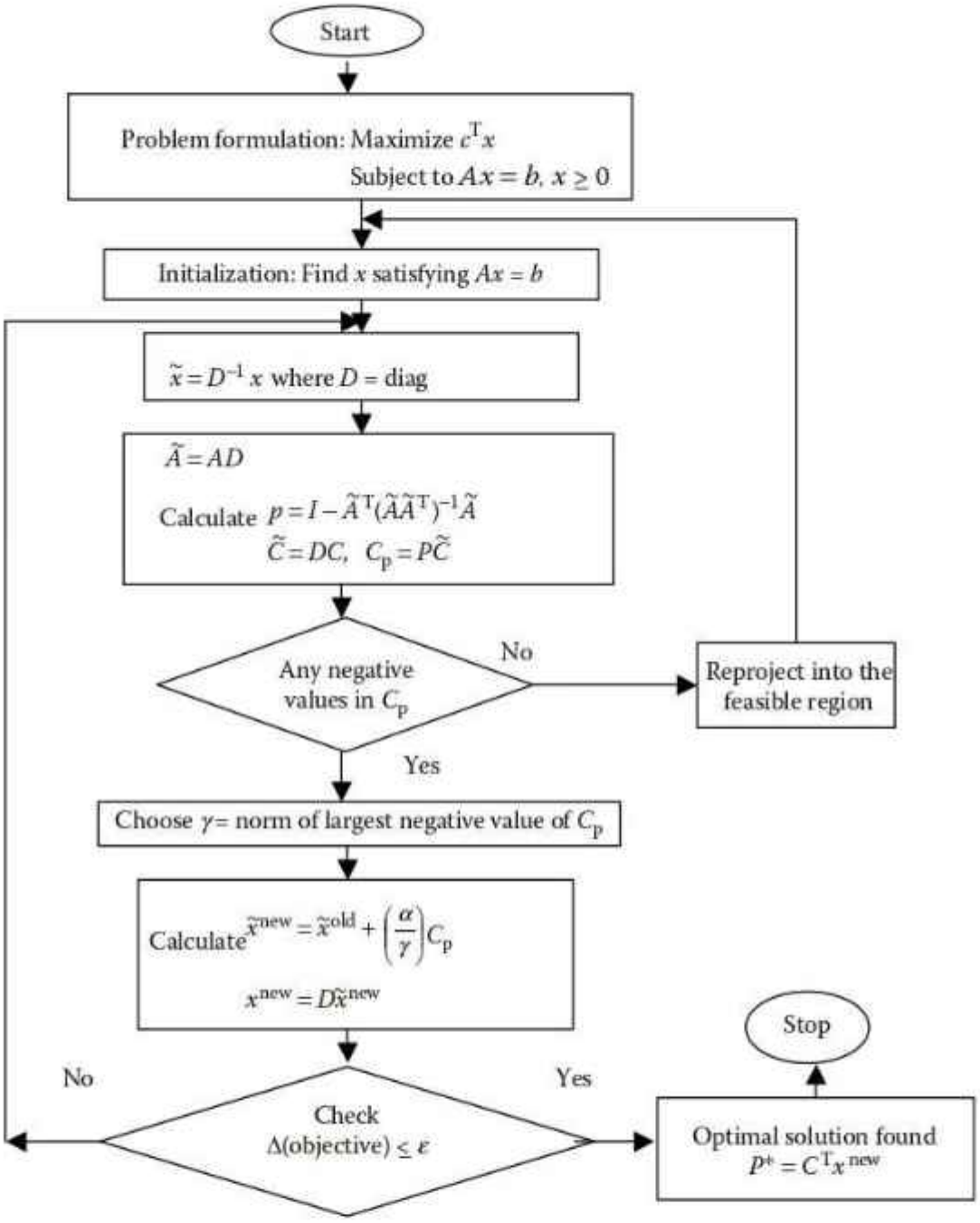


FIGURE 6.2
Algorithm for the IP method.

Multiplying both sides of Equation 6.25 from $k = N$ to $k = K$ gives

$$\frac{x_i^{K+1}}{x_i^N} = \sum_{k=N}^{k=K} \left[1 + \beta_i x_i^k u_i^k\right] > \sum_{k=N}^{k=K} \left[1 + \beta_i x_i^k \varepsilon\right] > 1. \tag{6.26}$$

Since $x_i^{K+1} > x_i^N > 0$ for any K,

$$x_i = \lim_{x \to \infty} x_i^{K+1} > 0 \tag{6.27}$$

as K approaches infinity which violates the fact that $x_i = 0$; therefore, $u_i \leq 0$ or $U \leq 0$ must be true.

Using $u \leq 0$ and $d_p = 0$ at x, we obtain $u^T y \leq 0$ and $d_p^T e = 0$, where y is any feasible solution and e is the n-vector with all components equal to one. Substituting Equations 6.17 and 6.18 yields

$$\begin{aligned} u^T y - d_p^T e &= u^T y - u^T De = u^T (y - x) \\ &= (a^T - w^T A)(y - x) \\ &= a^T y - a^T x - w^T b + w^T b \\ &= a^T y - a^T x \leq 0, \end{aligned} \tag{6.28}$$

which indicates that x is a maximum solution.

6.9 Extended Quadratic Programming Using IP Method

Extended quadratic programming using the IP method (EQIP) considered here is an extension of the LP version of the IP method developed during the project. The objective function, a quadratic form, is defined by

$$P = \frac{1}{2} x^T Q x + a^T x \tag{6.29}$$

subject to

$$Ax = b \quad \text{and} \quad x \geq 0, \tag{6.30}$$

where Q is any square and symmetric matrix. LP is a special case of the quadratic programming problem when $Q = 0$. The concept for solving the quadratic programming problem is similar to that of LP problems. It is again assumed that the problem has a bounded solution. With A being of full rank m with $m < n$, there are at least two feasible solutions to the problem. The same FI sequence generated below guarantees optimality within the feasible region for quadratic optimization problems.

In order to maintain the solution of the problem of each iteration within the interior feasible region, the algorithm requires the calculation of the initial starting interior feasible point $\tilde{X}^0$; that is, $\tilde{A}\tilde{X}^0 = \tilde{b}$ with $\tilde{x}_j^0 \geq 0$. The initial feasible point can be obtained by introducing the artificial variable x_S. the EQIP obtains the initial feasible point by using an auxiliary problem.

$$\text{Maximize } [-x_S] \tag{6.31}$$

Subject to

$$\begin{aligned} &\tilde{A}\tilde{X} + (\tilde{b} - \tilde{A}e)x_S = \tilde{b}, \\ &\tilde{x}_j \geq 0, \quad j = n+1, \ \ldots, n+2m, \quad x_S \geq 0. \end{aligned} \tag{6.32}$$

Clearly, any feasible solution of the original problem is a maximum solution $x_S = 0$ for the auxiliary problem and vice versa. Since the latter always has a feasible point at

$$\tilde{X} = e \quad \text{and} \quad x_S = 1,$$

one may use this point as the initial starting point to solve the auxiliary problem by the EQIP with $\tilde{Q} = 0$ to reach a maximum solution and thus obtain a feasible initial point for the original problem. The key point is that the direction vector d_x at each iteration k can be approximately calculated but maintain feasibility of

$$\tilde{X}^{k=1},$$

for example

$$\tilde{A}\tilde{X}^{k+1} = \tilde{b}$$

with

$$\tilde{x}_j^{k+1} \geq 0, \quad j+n+1, \ \ldots, n+2m.$$

A feasible direction, along with the objective function increases, is found, and then an approximate step length is determined to guarantee the new feasible solution, which is strictly better than the previous one. The stopping criteria are the relative changes in the objective function at iterations; that is

$$|P_{k+1} - P_k| \max\{1, |P_k|\} < \varepsilon_1, \tag{6.33}$$

or the relative changes in interior feasible solutions in iterations; that is

$$|\tilde{X}^{k+1} - \tilde{X}^k| < \varepsilon, \tag{6.34}$$

where P_{k+1} and P_k are defined as follows:

$$P_{k+t} = \frac{1}{2}(\tilde{X}^{K+1})^{\mathrm{T}}\tilde{Q}\tilde{X}^{K+1} + \tilde{a}^{\mathrm{T}}\tilde{X}^{K+1}, \tag{6.35}$$

$$P_k = \frac{1}{2}(\tilde{X}^K)^{\mathrm{T}}\tilde{Q}\tilde{X}^K + \tilde{a}^{\mathrm{T}}\tilde{X}^K. \tag{6.36}$$

The optimality condition is computed until the maximum is satisfied.

$$P_{k+1} = P_k + \frac{\left\| d_{\mathrm{P}}^k \right\|^4}{-2T}. \tag{6.37}$$

Detailed EQIP Algorithm

A detailed step-by-step description of how the EQIP algorithm solves a quadratic objective function subject to linear constraints is presented below:

Step 1. Identify the problem defined by maximizing:

$$P = \frac{1}{2} x^{\mathrm{T}} Q x + a^{\mathrm{T}} x, \quad \text{subject to } Ax = b$$
$$\text{With } x_i \geq 0, \quad \text{for } i = 1, 2, 3, \ldots, n$$

Step 2. For all i and x_{n+1} is an FI point of the auxiliary problem of maximizing: x_{n+1}
Subject to $Ax + (b - Ae)x_{n+1} = b$
where $e = [1, 1, \ldots, 1]^{\mathrm{T}}$, column vector, $\begin{bmatrix} x \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} e \\ 1 \end{bmatrix}$ is a feasible IP because $Ax + (b - Ae)x_{n+1} = Ae + b - Ae = b$.
Evidently, $x = e$ is a feasible* IP *for the diagonal problem $c = b - Ae = 0$ since $Ae = b$.

Step 3. Construct an updated $B = AD$ for corrected x, which is modified in such a way as to maximize:
$(-x_{n+1})$ in the auxiliary problem.
At the maximum, $-x_{n+1} = 0$ and hence
$Ax + (b - Ae)x_{n+1} = b \Rightarrow Ax + 0 = b$
Thus, x is a feasible IP of the original problem.

Step 4. MN solution for $By = v$ is designed to obtain
$y = B^{\mathrm{T}}(BB^{\mathrm{T}})^{\mathrm{T}} v$,
where BB^{T} is nonsingular.
The reason to assume $v = (x_{n+1})C$ is to assure feasibility to be explained in the next step.

Step 5. $\gamma = \mathrm{Min}[\gamma_i]$, $i^{n+1} = i^k(1 + \beta\zeta)$ for all i and $x_{n+1}^{k+1} = x_{n+1}^k(1 - \beta)$
for the auxiliary problem, we have from: $v = (b - Ae)x_{n+1}^k$

$$\begin{aligned} Ax^{k+1} + (b - Ae)x_{n+1}^{k+1} &= A(x^k + \beta Dy) + \beta_1(b - Ae)x_{n+1}^{k+1} + 0 = b \\ &= Ax^k + \gamma + \beta[By - v] \\ &= Ax^k + (b - Ae)x_{n+1}^k + 0 = b \end{aligned}$$

since $By = v$, thus x^{k+1} is feasible if the objective function is increased by
$\left(-x_{n+1}^{k+1}\right) - \left(-x_{n+1}^k\right) = \beta x_{n+1}^k > 0$, because $\beta > 0$ is to be chosen.

5(a): $\gamma+1>0$: $\gamma=\text{Min}\,[y_i]>-1\rightarrow 1+y_i>0$ and

$$x_i^{k+1}=x_i^k(1+y_i)>0,$$

since $x_i^k>0$

$$x_{n+1}^{k+1}=x_{n+1}^k(1+\beta)=x_{n+1}^k(1-1)=0$$

Because we choose $b=1$ for such a case. Therefore, x^{k+1} is a feasible IP of the original problem because

$$Ax^{k+1}+(b-Ae)x_{n+1}=Ax^{k+1}+0$$

5(b): $\gamma+1\leq 0$: $\beta=\frac{\alpha}{\gamma}>1$

$$1+\beta\xi=1\left(\frac{\alpha}{-\gamma}\right)y_i>1-1=0$$

$$1-\beta=1\left(\frac{\alpha}{-\gamma}\right)>1-\alpha>0$$

since $0<\alpha<1$; $\gamma\leq -1$, therefore x^{k+1} is an FI point of the auxiliary problem if x^k together with x_{n+1}^k is one.

Step 6. We exit from the auxiliary problem and now enter the original problem started with the FI point just obtained. For this purpose we construct r, $d=Dg$ and $B=AD$ as required.

Step 7. Solve for DG from $Bdg=v$ by using the minimum norm program with $v=Bd$.

Step 8. Replace zero and infinity, respectively, by
ε and M for computer practice.
Assume $M=10^6$ and $0.001\leq\varepsilon\leq 0.01$
8(a): $|dp_{ii}|\leq\varepsilon\rightarrow dp_i=0$ for all i and hence the optimally conditions are reached.
8(b): If a is not met, we will advance the FI point toward the limiting point according to: $x^k+\beta Ddp$, where $D=\text{diag}[x_1^k, x_2^k, \ldots, x_n^k]$
The scalar β must be chosen to ensure FI and also increase P. It will be chosen between β_1 and β_2 in step 2, but β_1 is fixed here as follows.
8(b) (i): $\gamma<0:\beta_1=(\frac{\alpha}{-\gamma})$ with $0<\alpha<1$ and
$\gamma=\min[d_{p_i}]$ such as β_1 makes $1+\beta_1 dp_1\geq(1-\frac{\alpha}{\beta})\alpha=1-\alpha>0$
Hence
$x_i^{k+1}=x_i^k(1+\beta_1 dp_1)>0$. Thus, x_i^{k+1} is a feasible IP is one
8(b) (ii): $\gamma\geq 0$: x_i^{k+1} is always interior for any $\beta_1>0$

We will choose $\beta_1 = M$ and leave β to be constrained by another β_2 to consider in step 9.

Step 9. It has been shown that the objective function between two consecutive points are related by

$$P_{k-1} = P_k + \beta \|dp\|^2 + \frac{1}{2}\beta^2 T$$

where $T = (Q\,dp)^{\mathrm{T}} Q(D\,dp)$, a scalar. As a function of β,

$$\beta = \beta_2 = \frac{\|dp\|^2}{-T}$$

If $T < 0$ and P_{k+1} increases monotonically with β for $T \geq 0$. Thus, one should choose $\beta = M$ for the latter case and β_2 for the former case. The scalar β_2 is fixed in this step for latter case.

Step 10. Problem does not have a finite maximum if $\beta_1 = M$ and $\beta_2 = M$ at the same time.

Step 11. β is chosen to be smaller one between β_1 and β_2 in order to guarantee that x^{k+1} maximize P_{k+1} and yet preserve FI point status with such β, we form the next point by

$$x_i^{k+1} = x_i^k + \beta x_i^k\, dP_i \quad \text{or} \quad x^{k+1} = x^k[1 + \beta D\, dP]$$

in matrix form as used in step 8.

Step 12. If $\beta = \beta_2$ was chosen, then no adjustment is needed, but we adjust the last point by choosing $\alpha = 1$ if $\beta = \beta_1$ was chosen. Let us consider x^{N+1} in S, that yields: $|dP(1)| \leq \varepsilon$ for a. Then instead of using fractional α, we use $\alpha = 1$ to generate x^N.

$$x^{N+1} = x^N + \frac{\alpha}{-\gamma} D\, dP = x + \frac{1}{\alpha}\Delta x$$

Eliminating x^N we have: $x = x^{N+1} + \delta\Delta x$
where $\delta = \frac{1}{\alpha} - 1 = 0.030927835$ for x 0.97 and $\Delta x = \frac{\alpha}{-\gamma} D\, dP$
which was computed in step 11.

Step 13. Current x is a solution because the optimal condition is met.

Step 14. No bounded solution exist because if $\beta_1 = M$ and $\beta_2 = M$ may be made infinity by some FI point (infinity).

6.10 Illustrative Examples

Example 6.10.1

Solve the constrained problem using the following:

Maximize $z = x_1 + 2x_2$

Subject to

$x_1 + x_2 + x_3 \leq 8$

$x_j \geq 0.$

1. IP method.
2. Graphical representation.

Based on the algorithm shown in Section 6.6 and following the flowchart in Figure 6.2, we can say $Z = x_1 + 2x_2 = C^t x \Rightarrow C^t = [1 \quad 2 \quad 0]$ as $x = [x_1, x_2, x_3]^t$

Subject to

$$Ax = b \Rightarrow A = [1 \quad 1 \quad 1].$$

We are going to take $\alpha = 0.7$, $\varepsilon = 0.1$.

First iteration

As an initial point, we start by $x = [1, 1, 2]^t$. Substitute in the objective function,

$$Z = C^t x = [1 \quad 2 \quad 0] \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 3.0$$

$$D = \text{diag}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\bar{x} = D^{-1}x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\bar{A} = AD = [1 \quad 1 \quad 1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = [1 \quad 1 \quad 2].$$

The projection area p,

$$P = I - \bar{A}^t(\bar{A}\bar{A}^t)^{-1}\bar{A} = \begin{bmatrix} 0.833 & -0.1667 & -0.333 \\ -0.1667 & 0.833 & -0.333 \\ -0.333 & -0.333 & 0.333 \end{bmatrix}$$

$$\tilde{C} = DC = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$C_p = P\tilde{C} = \begin{bmatrix} 0.5 \\ 1.5 \\ -1 \end{bmatrix}.$$

Then we can get the value of $\gamma = 1$.

$$\tilde{x}^{new} = \tilde{x}^{old} + \left(\frac{\alpha}{\gamma}\right) C_p = \begin{bmatrix} 1.35 \\ 2.05 \\ 0.3 \end{bmatrix}$$

$$x^{new} = D\tilde{x}^{new} = \begin{bmatrix} 1.35 \\ 2.05 \\ 0.6 \end{bmatrix}$$

$$Z^{new} = C'x^{new} = [1 \quad 2 \quad 0] \begin{bmatrix} 2.35 \\ 2.05 \\ 0.6 \end{bmatrix} = 5.45$$

$$\Delta \text{ objective} = Z^{new} - Z^{old} = 5.45 - 3.0 = 2.45 > \varepsilon.$$

Then we go to the second iteration.

Second iteration

$$x = [1.35, 2.05, 0.6]^t$$

$$D = \text{diag}(x) = \begin{bmatrix} 1.35 & 0 & 0 \\ 0 & 2.05 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}$$

$$\tilde{x} = D^{-1}x = \begin{bmatrix} 1.35 & 0 & 0 \\ 0 & 2.05 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}^{-1} \begin{bmatrix} 1.35 \\ 2.05 \\ 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} (1/1.35) & 0 & 0 \\ 0 & (1/2.05) & 0 \\ 0 & 0 & (1/0.6) \end{bmatrix} \begin{bmatrix} 1.35 \\ 2.05 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\tilde{A} = AD = [1 \quad 1 \quad 1] \begin{bmatrix} 1.35 & 0 & 0 \\ 0 & 2.05 & 0 \\ 0 & 0 & 0.6 \end{bmatrix} = [1.35 \quad 2.05 \quad 0.6].$$

The projection area p,

$$P = I - \tilde{A}^t(\tilde{A}\tilde{A}^t)^{-1}\tilde{A} = \begin{bmatrix} 0.7146 & -0.4334 & -0.1269 \\ -0.4334 & 0.3418 & -0.1926 \\ -0.1269 & -0.1926 & 0.9436 \end{bmatrix}$$

$$\bar{C} = DC = \begin{bmatrix} 1.35 & 0 & 0 \\ 0 & 2.05 & 0 \\ 0 & 0 & 0.6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.35 \\ 4.1 \\ 0 \end{bmatrix}$$

$$C_p = P\bar{C} = \begin{bmatrix} -0.8124 \\ 0.8163 \\ -0.9611 \end{bmatrix}.$$

Then we get the value of $\gamma = 0.9611$.

$$\bar{x}^{new} = \bar{x}^{old} + \left(\frac{\alpha}{\gamma}\right) C_p = \begin{bmatrix} 0.4083 \\ 0.5945 \\ 0.3 \end{bmatrix}$$

$$x^{new} = D\bar{x}^{new} = \begin{bmatrix} 0.5512 \\ 3.2688 \\ 0.18 \end{bmatrix}$$

$$Z^{new} = C^t x^{new} \; [1 \quad 2 \quad 0] \begin{bmatrix} 0.5512 \\ 3.2688 \\ 0.18 \end{bmatrix} = 7.0888$$

$$\Delta \text{ objective} = Z^{new} - Z^{old} = 7.0888 - 5.45 = 1.6388 > \varepsilon.$$

Then we go to the third iteration.

Third iteration

$$x = \begin{bmatrix} 0.5512 \\ 3.2688 \\ 0.18 \end{bmatrix} \quad \gamma = 0.5328$$

$$x^{new} = \begin{bmatrix} 0.1654 \\ 3.7383 \\ 0.0963 \end{bmatrix} \quad Z = 7.6421$$

$$\Delta \text{ objective} = Z^{new} - Z^{old} = 7.6421 - 7.0888 = 0.5533 > \varepsilon.$$

Then we go to the fourth iteration.

Fourth iteration

$$x = \begin{bmatrix} 0.1654 \\ 3.7383 \\ 0.0963 \end{bmatrix} \quad \gamma = 0.1923$$

$$x^{new} = \begin{bmatrix} 0.0661 \\ 3.905 \\ 0.0289 \end{bmatrix} \quad Z = 7.9577$$

$$\Delta \text{ objective} = Z^{\text{new}} - Z^{\text{old}} = 7.9577 - 7.6421 = 0.0816 < \varepsilon.$$

Then we can stop here with a result

$$x_1 = 0.0661, \quad x_2 = 3.9689, \quad x_3 = 0.0112, \quad \text{and} \quad Z = 7.9577.$$

Example 6.10.2

Consider the following problem

Maximize $z = 3x_1 + x_2$

Subject to

$x_1 + x_2 \leq 4$

$x_j \leq 0.$

Starting from the initial point (1,2) solve the problem using the IP algorithm.

Based on the algorithm shown in Section 6.6 and following the flowchart in Figure 6.2, we can say $Z = 3x_1 + x_2 = C^t x \Rightarrow C^t = [3 \quad 1]$ as $x = [x_1, x_2]^t$

Subject to $Ax = b \Rightarrow A = [1 \quad 1]$.

We are going to take $\alpha = 0.7$, $\varepsilon = 0.01$.

First iteration

As an initial point, we start with $x = [1, 2]^t$. Substitute in the objective function

$$Z = C^t x = [3 \quad 1] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5.0$$

$$D = \text{diag}(x) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\bar{x} = D^{-1}x = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{A} = AD = [1 \quad 1] \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = [1 \quad 2].$$

The projection area p

$$P = I - \bar{A}^t(\bar{A}\bar{A}^t)^{-1}\bar{A} = \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 0.2 \end{bmatrix}$$

$$\bar{C} = DC = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$C_p = P\bar{C} = \begin{bmatrix} 1.6 \\ -0.8 \end{bmatrix}.$$

Then we get the value of $\gamma = 0.8$.

$$\tilde{x}^{\text{new}} = \tilde{x}^{\text{old}} + \left(\frac{\alpha}{\gamma}\right) C_p = \begin{bmatrix} 2.4 \\ 0.3 \end{bmatrix}$$

$$x^{\text{new}} = D\tilde{x}^{\text{new}} = \begin{bmatrix} 2.4 \\ 0.6 \end{bmatrix}$$

$$Z^{\text{new}} = C^t x^{\text{new}} = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 2.4 \\ 0.6 \end{bmatrix} = 7.8$$

$$\Delta \text{ objective} = Z^{\text{new}} - Z^{\text{old}} = 7.8 - 5.0 = 2.8 > \varepsilon.$$

Then we go to the second iteration.

Second iteration

$$x = [2.4,\ 0.6]^t$$

$$D = \text{diag}(x) = \begin{bmatrix} 2.4 & 0 \\ 0 & 0.6 \end{bmatrix}$$

$$\tilde{x} = D^{-1}x = \begin{bmatrix} 2.4 & 0 \\ 0 & 0.6 \end{bmatrix}^{-1} \begin{bmatrix} 2.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\tilde{A} = AD = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2.4 & 0 \\ 0 & 0.6 \end{bmatrix} = \begin{bmatrix} 2.4 & 0.6 \end{bmatrix}.$$

The projection area P

$$P = I - \tilde{A}^t(\tilde{A}\tilde{A}^t)^{-1}\tilde{A} = \begin{bmatrix} 0.0588 & -0.2353 \\ -0.2353 & 0.9412 \end{bmatrix}$$

$$\tilde{C} = DC = \begin{bmatrix} 7.2 \\ 0.6 \end{bmatrix}$$

$$C_p = P\tilde{C} = \begin{bmatrix} 0.2824 \\ -1.1294 \end{bmatrix}.$$

Then we get the value of $\gamma = 1.1294$.

$$\tilde{x}^{\text{new}} = \tilde{x}^{\text{old}} + \left(\frac{\alpha}{\gamma}\right) C_p = \begin{bmatrix} 1.2471 \\ 0.0118 \end{bmatrix}$$

$$x^{\text{new}} = D\tilde{x}^{\text{new}} = \begin{bmatrix} 2.9929 \\ 0.0071 \end{bmatrix}$$

$$Z^{\text{new}} = C^t x^{\text{new}} = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 2.9929 \\ 0.0071 \end{bmatrix} = 8.9859$$

$$\Delta \text{ objective} = Z^{\text{new}} - Z^{\text{old}} = 8.9859 - 7.8 = 1.1859 > \varepsilon.$$

Then we go to the third iteration.

Third iteration

$$x = [2.9929, 0.0071]^t$$

$$D = \text{diag}(x) = \begin{bmatrix} 2.9929 & 0 \\ 0 & 0.0071 \end{bmatrix}$$

$$\tilde{x} = D^{-1}x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\tilde{A} = AD = [2.9929 \quad 0.0071].$$

The projection area P,

$$P = I - \tilde{A}^t(\tilde{A}\tilde{A}^t)^{-1}\tilde{A} = \begin{bmatrix} 0.0 & -0.0024 \\ -0.0024 & 1.0 \end{bmatrix}$$

$$\tilde{C} = DC = \begin{bmatrix} 8.9788 \\ 0.0071 \end{bmatrix}$$

$$C_p = P\tilde{C} = \begin{bmatrix} 0.00 \\ -0.0142 \end{bmatrix}.$$

Then we get the value of $\gamma = 0.0142$.

$$\tilde{x}^{\text{new}} = \tilde{x}^{\text{old}} + \left(\frac{\alpha}{\gamma}\right)C_p = \begin{bmatrix} 1.0017 \\ 0.2980 \end{bmatrix}$$

$$x^{\text{new}} = D\tilde{x}^{\text{new}} = \begin{bmatrix} 2.9979 \\ 0.0021 \end{bmatrix}$$

$$Z^{\text{new}} = C^t x^{\text{new}} = [3 \quad 1]\begin{bmatrix} 2.9979 \\ 0.0021 \end{bmatrix} = 8.9958$$

$$\Delta \text{ objective} = Z^{\text{new}} - Z^{\text{old}} = 8.9958 - 8.9859 = 0.0099 < \varepsilon.$$

Then, we stop up to the third iteration

$$x_1 = 2.9979, \quad x_2 = 0.0021, \quad \text{and} \quad Z = 8.9958.$$

Example 6.10.3

The following optimization problem demonstrates the primal affine-scaling algorithm.

Minimize $Z = 2x_1 + x_2 + 4x_3$

Subject to

$x_1 + x_2 + 2x_3 = 3$

$2x_1 + x_2 + 3x_3 = 5$

$x_i \geq 0 \quad (i = 1, 2, 3).$

First iteration

Let

$$e = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x^0 = \begin{bmatrix} 1.5 \\ 0.5 \\ 0.5 \end{bmatrix},$$

$$D_0 = \text{diag}(x^0) = \begin{bmatrix} 1.5 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.5 \end{bmatrix}.$$

Therefore, the dual estimate vector is

$$w^0 = (AD_0^2A^T)^{-1}AD_0^2c = \begin{bmatrix} 0.8947 \\ 0.5789 \end{bmatrix},$$

and the reduced cost coefficient is

$$r^0 = \begin{bmatrix} -0.0526 \\ -0.4737 \\ 0.4737 \end{bmatrix}.$$

For the optimality check, we calculate

$$e^T = D_0r^0 = -0.0789 \quad \text{and} \quad d_y^0 = \begin{bmatrix} 0.0789 \\ 0.2368 \\ -0.2368 \end{bmatrix}.$$

The optimality condition is not satisfied but the problem is not unbounded; therefore

$$\beta_0 = 4.1807.$$

The update on the primal variable is

$$x^1 = x^0 + \beta_0 D_0 d_y^0 = \begin{bmatrix} 1.9951 \\ 0.9951 \\ 0.0049 \end{bmatrix}.$$

Second iteration

$$D_1 = \begin{bmatrix} 1.9951 & 0.0000 & 0.0000 \\ 0.0000 & 0.9951 & 0.0000 \\ 0.0000 & 0.0000 & 0.0049 \end{bmatrix}$$

$$w^1 = \begin{bmatrix} 0 \\ 1.0 \end{bmatrix} \quad \text{and} \quad r^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

For the optimality checking, we compute

$$e^T D_1 r^1 = 0.0049$$

$$d_y^1 = \begin{bmatrix} 0.0 \\ 0.0 \\ -0.0049 \end{bmatrix},$$

which implies that the problem is bounded.

Therefore, we compute $\beta_1 = 202.0408$ and update the primal variable to get

$$x^2 = \begin{bmatrix} 2.0 \\ 1.0 \\ 0.0 \end{bmatrix} = x^*,$$

which yields an optimal value of objective value $Z^* = 5$.

Example 6.10.4

The following optimization problem demonstrates the dual affine-scaling algorithm.

Maximize $Z = 15x_1 + 15x_2$

Subject to

$1.5x_1 + 5x_1 = -3.0$

$-1.5x_1 + 1.5x_2 + 5x_2 = 1.5$

$1.5x_3 + s_3 = 0$

$1.5x_4 + s_4 = 0$

$s_i \geq 0 \quad (i = 1, 2, 3, 4).$

First iteration

$$x^0 = \begin{bmatrix} -3 \\ -3 \end{bmatrix}, \quad s^0 = \begin{bmatrix} 1.5 \\ 1.5 \\ 4.5 \\ 4.5 \end{bmatrix},$$

$$W^0 = \text{diag}(s^0) = \begin{bmatrix} 1.5 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0 & 4.5 & 0 \\ 0 & 0 & 0 & 4.5 \end{bmatrix}.$$

Therefore, the direction of translation is

$$d_x^0 = (A^t W_0^{-2} A)^{-1} c = \begin{bmatrix} 23.5321 \\ 34.6789 \end{bmatrix},$$

$$d_s^0 = -A^T d_x^0 = \begin{bmatrix} -35.2982 \\ -16.7202 \\ -35.2982 \\ -52.0183 \end{bmatrix},$$

$$y^0 = \begin{bmatrix} 15.6881 \\ 7.4312 \\ 1.7431 \\ 2.5688 \end{bmatrix}.$$

For the optimality check, we calculate

$$b^T y^0 - c^T X^0 = 46.2385.$$

The optimality condition is not satisfied but the problem is not unbounded; therefore

$$\beta_0 = 0.0421.$$

The update on the primal variable is

$$x^1 = x^0 + \beta_0 d_x^0 = \begin{bmatrix} -2.01 \\ -1.5411 \end{bmatrix}$$

$$s^1 = \begin{bmatrix} 0.015 \\ 0.796 \\ 3.015 \\ 2.312 \end{bmatrix}.$$

Second iteration

$$d_x^1 (A^T W_1^{-2} A)^{-1} c = \begin{bmatrix} 0.0028 \\ 3.7838 \end{bmatrix}$$

$$d_s^1 = -A^T d_x^1 = \begin{bmatrix} -0.0043 \\ -5.6714 \\ -0.0043 \\ -5.6757 \end{bmatrix}$$

$$y^1 = \begin{bmatrix} 18.9374 \\ 8.9378 \\ 0.0005 \\ 1.0622 \end{bmatrix}.$$

For the optimality check, we calculate

$$b^T y^1 - c^T X^1 = 0.3918.$$

The optimality condition is not satisfied but the problem is not unbounded; therefore

$$\beta_0 = 0.1391.$$

The update on the primal variable is

$$x^2 = x^1 + \beta_1 d_x^1 = \begin{bmatrix} -2.0096 \\ -1.0149 \end{bmatrix}$$

$$s^2 = \begin{bmatrix} 0.0144 \\ 0.0079 \\ 3.0144 \\ 1.5224 \end{bmatrix}.$$

The reader may carry out more iterations and verify that the optimal value is assumed at $X^* = (-2, -1)^T$ and $s^* = (0, 0, 3, 1.5)^T$.

Example 6.10.5

Using IP method, solve the following LP problem:

$$\begin{aligned} \text{Maximize} \quad & 3x_1 + 3x_2 - x_3 \\ \text{subject to} \quad & 2x_1 - 3x_2 + x_3 = 0 \\ & x_1 + x_2 + x_3 = 1 \\ & \forall j \in \{1,2,3\} \end{aligned}$$

Solution

The problem is formulated in standard form as

$$\text{Max } z = 3x_1 + 3x_2 - x_3 = C^T x$$

$$C^T = [3 \quad 3 \quad -1] \text{ as } x = [x_1, x_2, x_3]^T$$

$$\text{subject to } Ax = b \Rightarrow A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

And using $\alpha = 0.7$ and $\varepsilon = 0.1$

As an initial point, we start by $z = C^T x = [3 \quad 3 \quad -1] \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 4.0$, $x = [1, 1, 2]^T$

$$z = C^T x = [3 \quad 3 \quad -1] \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 4.0, \quad D = \text{Diag } x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\tilde{x} = D^{-1}x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\tilde{A} = AD = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

The projection matrix is calculated as

$$P = I - \tilde{A}^{\mathrm{T}}(\tilde{A}\tilde{A}^{\mathrm{T}})^{-1}\tilde{A}$$

$$\Rightarrow P = \begin{bmatrix} 0.0025 & -0.0045 & 0 \\ 0 & -0.023 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{c} = DC = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -0.006 \\ -0.069 \\ -2 \end{bmatrix}$$

$$\therefore \gamma = |-2| = 2, \quad \alpha = 0.7$$

$$\therefore \tilde{x}_{\text{new}} = \tilde{x}_{\text{old}} + \left(\frac{\alpha}{\gamma}\right) C_{\text{p}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{0.7}{2} \begin{bmatrix} -0.006 \\ -0.069 \\ -2 \end{bmatrix} = \begin{bmatrix} -0.0021 \\ 0.97585 \\ -0.7 \end{bmatrix}$$

$$x_{\text{new}} = D\tilde{x}_{\text{new}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -0.0021 \\ 0.97585 \\ -0.7 \end{bmatrix} = \begin{bmatrix} -0.0021 \\ 0.97585 \\ -1.4 \end{bmatrix}$$

$$\text{Profit}^{\text{new}} = C^{\mathrm{T}}x_{\text{new}} = [3 \quad 3 \quad -1] \begin{bmatrix} -0.0021 \\ 0.97585 \\ 1.4 \end{bmatrix} = 4.32125$$

After the second iteration, the value of x was found to be $x^* = \begin{bmatrix} -0.021 \\ 0.9758 \\ -0.7 \end{bmatrix}$ and the profit $P^* = 3.621$.

Example 6.10.6

Recall the linear model

Maximize $f(x) = c^{\mathrm{T}}x$

Subject to $Ax = b$

From the given problem, the following vectors and matrices are identified:

$$x = [x_1 \quad x_2 \quad x_3]^{\mathrm{T}}, \quad c^{\mathrm{T}} = [2 \quad 5 \quad 7], \quad A = [1 \quad 2 \quad 3], \quad \text{and} \quad b = 6.$$

First iteration
Initialize: $x^{(0)} = [1 \quad 1 \quad 1]^T$ such that $Ax = b$
Choosing: $\alpha = 0.95$, where $0 < \alpha < 1$, $\varepsilon = 10^{-3}$
Now, the scaled gradient vector is $\tilde{c} = DC = [2 \quad 5 \quad 7]^T$
And from the problem, $A = [1 \quad 2 \quad 3]$ such that $\tilde{A} = AD = AI = A$
The projection matrix is calculated as

$$P = I - \tilde{A}^T(\tilde{A}\tilde{A}^T)^{-1}\tilde{A} = \begin{bmatrix} 0.9286 & -0.1429 & -0.2143 \\ -0.1429 & 0.7143 & -0.4286 \\ -0.2143 & -0.4286 & 0.3571 \end{bmatrix}$$

The projection gradient is

$$C_p = P\tilde{c} = \begin{bmatrix} 0.9286 & -0.1429 & -0.2143 \\ -0.1429 & 0.7143 & -0.4286 \\ -0.2143 & -0.4286 & 0.3571 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -0.3571 \\ +0.2857 \\ -0.0714 \end{bmatrix}$$

Scalar value $\gamma = |\text{Min } C_{p_i}| = |-0.3571| = 0.3571$ for $C_{p_i} < 0$
Updating the scaled solution vector yields:

$$\tilde{x}^{new} = \tilde{x}^{old} + \frac{\alpha C_p}{\gamma}$$

$$\tilde{x}^{new} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{0.95}{0.3571} \begin{bmatrix} -0.3571 \\ 0.28570 \\ -0.07140 \end{bmatrix} = \begin{bmatrix} 0.0500 \\ 1.7600 \\ 0.8100 \end{bmatrix}$$

Hence, the scaled solution vector is

$$x^{new} = D\tilde{x}^{old} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.0500 \\ 1.7600 \\ 0.8100 \end{bmatrix} = \begin{bmatrix} 0.0500 \\ 1.7600 \\ 0.8100 \end{bmatrix}$$

The value of the objective function at this stage is $f(x^{new}) = 14.5700$

After another iteration, the optimal solution was found to be $x^* = \begin{bmatrix} 0.0461 \\ 2.9162 \\ 0.0405 \end{bmatrix}$

and $f^* \approx 14.9567$.

6.11 Conclusions

Variants of IP algorithms were presented. These variants included work by Karmarkar, projection, offline-scaling, and the primal-affine algorithm. These methods were shown in Sections 6.2 through 6.5.

In Section 6.6, the barrier algorithm was presented, where a barrier-function tests hit inequality constraint methods by creating a barrier function which is a combination of the original objective function and a weighted sum

of functions with a positive singularity at the boundary. The formulation and algorithm were presented in this section.

In Section 6.7, an extended IP for the LP problem was presented and a discussion of the possible interior sequence was presented in Section 6.8 where the optimality conditions and start and termination of the recursive process were explained. In Section 6.9, an extended quadratic programming algorithm for solving quadratic optimization problems was presented.

6.12 Problem Set

PROBLEM 6.12.1

Solve the unconstrained problem:

$$\text{Minimize } z = \frac{1}{3}x_1^2 + \frac{1}{2}x_2^2 - x_1x_2 - 2x_1,$$

1. Using the IP method
2. Any other method

PROBLEM 6.12.2

Solve the following problem using the quadratic IP method:

$$\text{Minimize } z = 2x_1^2 + 3x_2^2 + 5x_3^2 + x_1 + 2x_2 - 3x_3$$

Subject to

$$x_1 + x_2 = 5$$

$$x_1 + x_3 = 10$$

$$x_j \geq 0.$$

PROBLEM 6.12.3

Consider the following problem.

$$\text{Maximize } z = 2x_1 + 5x_2 + 7x_3$$

Subject to

$$x_1 + 2x_2 + 3x_3 = 6$$

$$x_j \geq 0.$$

1. Graph the feasible region.
2. Find the gradient of the objective function and then find the projected gradient onto the feasible region.

3. Starting from initial trial solution (1, 1, 1) perform two iterations of the IP algorithm.
4. Perform eight additional iterations.

PROBLEM 6.12.4

Consider the following problem.

Maximize $z = -x_1 - x_2$

Subject to

$x_1 + x_2 \leq 8$

$x_2 \geq 3$

$-x_1 + x_2 \leq 2$

$x_j \geq 0.$

1. Solve this problem graphically.
2. Use the dual simplex method to solve this problem.
3. Trace graphically the path taken by the dual simplex method.
4. Solve this problem using the IP algorithm.

PROBLEM 6.12.5

Maximize $3x_1 + 3x_2 - x_3$

s.t. $2x_1 - 3x_2 + x_3 = 0$

$x_1 + x_2 + x_3 = 1$

PROBLEM 6.12.6

Maximize $4x_1 + 3x_2$

s.t. $3x_1 + 10x_2 \leq 8$

$4x_1 + 2x_2 \leq 15$

PROBLEM 6.12.7

Maximize $10x_1 + 4x_2 + 3x_3$

s.t. $x_1 - x_3 \leq 0$

$x_1 + x_2 \leq 2$

$-x_1 + x_2 + 3x_3 \leq 4$

$x_1, x_2, x_3 \geq 0$

PROBLEM 6.12.8

Maximize $2x_1 + 3x_2 + 4x_3$

Subject to

$x_1 + x_2 + x_3 \geq 5$

$x_1 + 2x_2 = 7$

$5x_1 - 2x_2 + 3x_3 \leq 9$

$x_1, x_2, x_3 \geq 0$

PROBLEM 6.12.9

Maximize $z = 3x_1 + 2x_2$

Subject to

$x_1 + 2x_2 + x_3 = 6$

$2x_1 + x_2 + x_4 = 8$

$-x_1 + x_2 + x_5 = 1$

$x_2 + x_6 = 2$

$x_1, x_2, \ldots, x_6 \geq 0$

References

1. Alder, I., Karmarkar, N., Resende, M. G. C., and Veiga, G., An implementation of Karmarkar's algorithm for linear programming, Working Paper, Operations Research Center, University of California, Berkeley, 1986 (also in *Mathematical Programming*, 44).
2. Alder, I., Resende, M. G. C., Veiga, G., and Karmarkar, N., An implementation of Karmarkar's algorithm, *ORSA Journal on Computing*, 1(2), 1989, 84–106.
3. Anstreicher, K. M. A., Monotonic projective algorithm for fractional linear programming, *Algorithmica*, 1, 1986, 483–498.
4. Barnes, E. R., A variation of Karmarkar's algorithm for solving linear programming problems, *Mathematical Programming 1986 for Computing Projections*, 13th International Mathematical Programming Symposium, Tokyo, August 1988.
5. Carpenter, J., Contribution a l'Etude du Dispatch Economique, *Bulletin de la Société Francaise des Electriciens*, 3, 1962, 431–447.
6. Dommel, H. W. and Tinney, W. F., Optimal power flow solutions, *IEEE Transactions on Power Apparatus and Systems*, 87, 1968, 1866–1878.
7. Galliana, F. D. and Hunneault, M., A survey of the optimal power flow literature, *IEEE Transactions on Power Systems*, 6, 1991, 762–770.
8. Karmarkar, N., New polynomial-time algorithm for linear programming, *Combinatorica*, 4, 1984, 373–397.

9. Karmarkar, N., *A New Polynomial-time Algorithm for Linear Programming*, Combinatorica, 1984, 373–395.
10. Kojima, M., Determining basic variables of optimal solutions in Karmarkar's new LP algorithm, *Algorithmica*, 1, 1986, 499–517.
11. Kozlov, A., The Karmarkar algorithm: Is it for real? *SIAM News*, 18(6), 1987, 1–4.
12. McShane, K. A., Monma, C. L., and Shanno, D., An implementation of a primal–dual interior point method for linear programming, *ORSA J. Computing*, Spring 1989, 70–83.
13. Meggido, N., On finding primal- and dual-optimal bases, Research Report, RJ 6328 (61997), IBM, Yorktown Heights, New York, 1988.
14. Momoh, J. A., Application of quadratic interior point algorithm to optimal power flow, EPRI Final Report RP 2473-36 II, Howard University, March, 1992.
15. Momoh, J. A., Austin, R., and Adapa, R., Feasibility of interior point method for VAR planning, accepted for publication in *IEEE SMC*, 1993.
16. Momoh, J. A., Guo, S. X., Ogbuobiri, E. C., and Adapa, R., The quadratic interior point method solving power system security-constrained optimization problems, Paper No. 93, SM 4T7-BC, Canada, July 18–22, 1993.
17. Ponnambalam, K., New starting and stopping procedures for the dual affine method, Working Paper, Department of Civil Engineering, University of Waterloo, Waterloo, Ontario, Canada, 1988.
18. Reid, G. F. and Hasdorf, L., Economic dispatch using quadratic programming, *IEEE Transactions on Power Apparatus and Systems*, PAS-92, 1973, 2017–2023.
19. Sun, D. I., Ashely, B. B., Hughes, A., and Tinney, W. F., Optimal power flow by Newton method, *IEEE Transactions on Apparatus and Systems*, PAS-103, 1984, 2864–2880.
20. Vanderbei, R. J., Meketon, M. S., and Freedman, B. A., A modification of Karmarkar's linear programming algorithm, *Algorithmica*, 1, 1986, 395–409.
21. Vannelli, A., An adaptation of the interior point method for solving the global routine problem, *IEEE Transactions on Computer-Aided Design*, 10(2) 1991, 193–203.
22. Ye, Y. and Kojima, M., Recovering optimal dual solutions in Karmarkar's polynomial-time algorithm for linear programming, *Mathematical Programming*, 39, 1987, 307–318.
23. Kojima, M., Determining basic variables of optimal solutions in Karmarkar's new LP algorithm, *Algorithmica*, 1(4), 1986, 449–517.
24. Montiero, R. C. and Adler, I., Interior path following primal–dual algorithms. Part I: Linear programming, *Mathematical Programming*, 44, 1989, 27–42.
25. Dantzig, G. F., *Linear Programming and Extensions*, Princeton University Press, Princeton, NJ, 1963.
26. Lustig, I. J., Marsten, R. E., and Shanno, D. F., Interior point methods for linear programming: computational state of the art, *ORSA Journal on Computing*, 6(1), 1994, 1–14.
27. Liu, M. Tso, S. K., and Cheng, Y., An extended nonlinear primal–dual interior-point algorithm for reactive-power optimization of large-scale power systems with discrete control variables, *IEEE Transactions on Power Systems*, 17(4), 2002, 982–991.

7

Nonlinear Programming

7.1 Introduction

While linear programming has found numerous practical applications, the assumptions of proportionality, additivity, and other forms of nonlinearity are common in many engineering applications. Most often the sources of nonlinearity are the physical process and the associated engineering principles that are not amenable to linearization. Even though there are linearization schemes, they are subject to large errors in representing the phenomenon.

Nonlinear programming (NLP) aims to solve optimization problems involving a nonlinear objective and constraint functions [1–4,10–13]. The constraints may consist of equality and inequality forms. The inequalities may be specified by two bounds: bounded below and bounded above. There is no generalized approach to solve the NLP problem and a particular algorithm is usually employed to solve the specified type of problem. In other words, it is different from the simplex method which can be applied to any LP problem. However, two methods, namely, sensitivity and barrier, are considered to be quite generalized to be able to successfully solve the NLP. These methods are discussed in detail in this chapter.

Theorems on necessary and sufficient conditions are given for extremizing unconstrained functions and optimizing constrained functions. In conjunction with the necessary condition, the well-known Kuhn–Tucker (K–T) conditions are treated. Finally, based on the K–T conditions, sensitivity and barrier methods are developed for a general approach to solve the NLP problems. The methods are designed for solving NLP involving large numbers of variables such as the power system.

7.2 Classification of NLP Problems

7.2.1 NLP Problems with Nonlinear Objective Function and Linear Constraints

This is a relatively simple problem with nonlinearity limited only to the objective function. The search space is similar to that of the linear

programming problem and the solution methods are developed as extensions to the simplex method.

7.2.2 Quadratic Programming

This is a special case of the former where the objective function is quadratic (i.e., involving the square or cross-product of one or more variables). Many algorithms have been developed with the additional assumption that the objective function is convex, which is a direct extension of the simplex method. Apart from being a very common form for many important problems, quadratic programming (QP) is also very important because many of the problems in Section 7.3.1 are often solved as a series of QP or sequential quadratic programming (SQP) problems.

Objective form:

$$f(x) = \sum c_j x_j + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} x_i x_j.$$

Subject to

$$\sum a_{ij} x_j \leq b_i$$

$$x_j \geq 0,$$

where

c_j, a_{ij} and b_i and a_{ij} are assumed to be known

$a_{ij} = a_{ji}$ are symmetrical and hence $= a_{ij} = \frac{1}{2}[a_{ij} + a_{ji}]$

7.2.3 Convex Programming

Convex programming arises out of the assumptions of convexity of the objective and constraint functions. Under these assumptions, it can encompass both foregoing problems. The major point of emphasis is that the local optimal point is necessarily for the global optimum under these assumptions.

7.2.4 Separable Programming

Separable programming is a special class of convex programming with the additional assumption that all objective and constraint functions are separable functions; that is, the function can be expressed as a sum of the functions of the individual variables. For example, if $f(x)$ is a separable function it can be expressed as

$$f(x) = \sum_{j=1}^{n} f_j(x_j),$$

where each, $f_j(x_j)$ includes a term involving x_j only.

Subject to

$$\sum g_{ij}(x_j) \leq 0.$$

These problems are called separable programming because the decisive variable approximate separate one in each function g_{ij} in the constraints and one in the each objective function.

7.3 Sensitivity Method for Solving NLP Variables

For simplicity, we hereafter use $i \leq p$ to mean $i = 1, 2, \ldots, p$, $i > p$ to mean $i = P + 1, p + 2, \ldots, m$, and all i to mean $i = 1, 2, \ldots, m$. Let y be an n-vector and $f(y)$ together with $f_i(y)$ be scalar functions for all i. Then, the NLP problem is defined to

Minimize $f(y)$

Subject to

$$C_i \leq f_i(y) \leq D_i \quad \text{for all } i, \tag{7.1}$$

where

$$C_i = D_i \quad \text{for } i \leq p \quad \text{but } C_i < D_i \quad \text{for } i > p.$$

There are p equality constraints and $m - p$ inequality constraints which are bounded by C_i and D_i for $i > p$. The number of p is less than or equal to n but m may be greater than n. By $C_i = -\infty$ and $D_i = \infty$, we mean that the inequality constraint is bounded above and below, respectively. Any constraint with one bound is thus a special case of the inequality constraint.

The constraints are denoted by the equality form

$$f_i(y) = k_i \tag{7.2}$$

for all i where $K_i = D_i$, when $i \leq p$ and $C_i \leq K_i \leq D_i$, when $i > p$. In matrix form, they are denoted collectively by

$$F(y) = K, \tag{7.3}$$

where

$F(y) = [f_1(y), f_2(y), \ldots, f_m(y)]^T$
$K = [K_1, K_2, \ldots, K_m]^T$

The Lagrange function for this problem is defined as before by the scalar function

$$L(y,\lambda) = f(y) + \lambda F(y), \tag{7.4}$$

where $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_m]$.

The Lagrange function is assumed to be continuous up to the first partial derivatives at $y = x$ which is a minimum as considered in the theorem.

Extended K–T conditions are considered here to cover constraints bounded below and above. The conditions can be stated in theorem form as follows.

THEOREM 7.3.1

If x is a solution for the NLP problem, then it is necessary that $L_x(x,\lambda) = 0$ and one of the following conditions be satisfied for all $i > p$.

(a) $\lambda_i = 0$ *when* $C_i < f_i(x) < D_i$.

(b) $\lambda_i \geq 0$ *when* $f_i(x) = D_i$.

(c) $\lambda_i \leq 0$ *when* $f_i(x) = C_i$.

7.3.1 Procedure for Solving the NLP Problem

1. Use $L_x(x,\lambda) = 0$ and the equality constraints to find x for cases a, b, and c.
2. Find the smallest $f(x)$ among the three possible x obtained in step 1.
3. Use $T = L_{xx}(x,\lambda) + \beta F_x^{\mathrm{T}}(x) F_x(x) > 0$ for some $\beta \geq 0$ to test sufficiency for x determined in step 2.

The conditions as imposed in the theorem are called here the Extended Kuhn–Tucker (EKT) conditions. They suggest that one can predict the changes of $f(x)$ due to variations of K if λ is known. This fact is utilized in the method to approach the EKT conditions. Note that there may exist multiple sets of EKT conditions.

Transpose Equation 7.4 and then use the column vector z to denote λ^{T} and U to denote L_x^{T}:

$$U(x,z) = f_x^{\mathrm{T}}(x) + F_x^{\mathrm{T}}(x) z = 0, \tag{7.5}$$

where $z_i = \lambda_i$ for all i. Since Equation 7.3 must be satisfied for $y = x$, we have

$$F(x) = K, \tag{7.6}$$

where $K_i = C_i = D_i$ for $i \leq p$ but is uncertain for $i > p$. The method uses a process that adjusts K_i in $C_i \leq K_i \leq D_i$ for $i > p$ to decrease $f(x)$ and keep Equations 7.4 and 7.6 satisfied.

Consider some x, z, and K that satisfy Equations 7.5 and 7.6 at one step and $x + \Delta x$, $z + \Delta z$, and $K + \Delta K$ at the next. Then, it follows that

$$U(x + \Delta x,\, z + \Delta z) = f_x^{\mathrm{T}}(x + \Delta x) + F_x^{\mathrm{T}}(x + \Delta x)(z + \Delta z) = 0$$

and $F(x + \Delta x) = K + \Delta K$.

The first-order approximation of the above equations is

$$U(x,z) + U_x(x,z)\Delta x + U_z(x,z)\Delta z = 0 \tag{7.7}$$

and

$$F^{\text{new}}(x) = F^{\text{old}}(x) + F_x(x)\Delta x = K + \Delta K. \tag{7.8}$$

The partial derivatives of $U(x, z)$ can be found from Equation 7.5 as

$$S = U_x(x,z) = f_{xx}(x) + \sum_{i=1}^{m} \lambda_i f_{ixx}(x) \tag{7.9}$$

and

$$U_z(x,z) = F_x^{\mathrm{T}}(x),$$

where f_{xx} and f_{ixx} are, respectively, the second derivatives of $f(x)$ and $f_i(x)$ with respect to x; all of them are assumed to exist.

For simplicity, the augments x and z are omitted here and Equations 7.7 and 7.8 are combined in a matrix form

$$Ay = b, \tag{7.10}$$

where

$$A = \begin{bmatrix} S & F_x^{\mathrm{T}} \\ F_x & 0 \end{bmatrix}, \quad y = \begin{bmatrix} \Delta x \\ \Delta z \end{bmatrix}, \quad b = \begin{bmatrix} -U \\ \Delta K \end{bmatrix}.$$

Note that the condition $U = 0$ may not be true due to first-order approximation but $F = K$ is true since K is calculated from Equation 7.6. Inclusion of U in the vector b would force U to be zero at the next step if it were not zero at the present one. Equation 7.10 shows that any two of the increments may be determined if the third one is specified. However, the change of K not only relates to the constraints but correlates with the objective function as evidenced by Equation 7.6. It is for this reason that ΔK is chosen to be the independent variable. As mentioned earlier, the basic rule of adjusting ΔK is to decrease $f(x)$ without violating the constraints.

The matrix A of Equation 7.10 may not be inverted at each step of the process; this is always the case for $m > n$. We are seeking here the least square solution with minimum norm (LSMN) for Δx and Δz. The solution always exists and is unique as long as A is not a null matrix. Moreover, it reduces automatically to the exact solution if A is nonsingular.

Expression for EKT Conditions (Sensitivity Method)

As it is, the EKT conditions are not suitable for application to a computer-based solution. An alternative expression is sought here for practical applications.

Consider an m-vector J with component J_i $(i=1, 2, \ldots, m)$ defined by

$$T = \min[\lambda_i, (D_i - K_i)] \tag{7.11}$$

and

$$J_i = \max[T, (C_i - K_i)], \tag{7.12}$$

for all $i=1, 2, \ldots, m$. It can be concluded that a set of EKT conditions is satisfied if and only if $U=0$ and $J=0$. This can be shown as follows:

1. *Feasibility* $K_i > D_i$: $J_i=0$ from Equation 7.12 requires that $T=0$ which cannot take place in Equation 7.11 since $D_i - K_i < 0$.

   $K_i < C_i$: $J_i=0$ from Equation 7.12 cannot occur since $C_i - K_i > 0$. Therefore, $C_i \leq K_i \leq D_i$ must hold when $J_i=0$.
2. *Optimality* $U=0$ is imposed in the sensitivity method. This fulfills the first part of the EKT conditions: $L_x=0$.

$C_i < K_i < D_i$: $J_i=0$ from Equation 7.12 requires that $T=0$ and hence $\lambda_i=0$ must hold in Equation 7.11. Conversely, if $\lambda_i=0$, then $T=0$ in Equation 7.11 and hence $J_i=0$ results from Equation 7.12.

$K_i=D_i$: $J_i=0$ from Equation 7.12 requires that $T=0$ and hence $\lambda_i \geq 0$ must hold in Equation 7.11. Conversely, if $\lambda_i \geq 0$, then $T=0$ in Equation 7.11 and hence $J_i=0$ results from Equation 7.12.

$K_i=C_i$: $J_i=0$ from Equation 7.12 requires that $T \leq 0$ and hence $\lambda_i \leq 0$ must hold in Equation 7.11. Conversely, if $\lambda_i \leq 0$, then $T \leq 0$ in Equation 7.11 and hence $J_i=0$ results from Equation 7.12.

It follows therefore that $J=0$ and $U=0$ are both necessary and sufficient to reach a set of EKT conditions. These conditions are used as the criteria for termination of the method. The process involved in the method may start with any guessed values of x and λ. However, different initial values may lead to different sets of EKT conditions and even divergence.

Then to compute ΔK for the algorithm:

1. Adjustment of ΔK

Consider a change Δx about a known x. The second-order approximation of the objective function can be written as

$$f(x + \Delta x) = f(x) + f_x \Delta x + \frac{1}{2} \Delta x^T f_{xx} \Delta x, \tag{7.13}$$

where the partial derivatives are evaluated at x. It is intended to reduce Equation 7.13 by a proper adjustment of ΔK. To this end, we assume that $\Delta K = qJ$, where J is determined from Equations 7.11 and 7.12. The increments Δx and Δz caused by ΔK satisfy Equation 7.10:

$$A\begin{bmatrix}\Delta x\\ \Delta z\end{bmatrix} = \begin{bmatrix}-U\\ qJ\end{bmatrix}, \tag{7.14}$$

which consists of

$$S\Delta x + F_x^{\mathrm{T}}\Delta x = -U$$

and $F_x\Delta x = \Delta K = qJ$.

We define vectors u and v in such a way that $\Delta x = qu$ and $F_x^{\mathrm{T}}(\Delta z - qv) = (q-1)U$ where the last equation is satisfied in the sense of LSMN [13]. Then, by eliminating Δx and Δz, we obtain from Equation 7.14 that

$$A\begin{bmatrix}u\\ v\end{bmatrix} = \begin{bmatrix}-u\\ J\end{bmatrix}. \tag{7.15}$$

Substitution of $\Delta x = qu$ into Equation 7.13 gives

$$f(x+\Delta x) = f(x) + qf_x u + \frac{1}{2}q^2 N,$$

where

$$N = u^{\mathrm{T}} f_{xx} u. \tag{7.16}$$

The minimum of $f(x+\Delta x)$ for positive q occurs at

$$q = -\frac{f_x u}{N} \tag{7.17}$$

if $f_x u < 0$ and $N > 0$. We choose q as given by Equation 7.17 only when $f_x u < 0$ and $N > 0$, and choose $\Delta K = J$ or $q = 1$ otherwise. For q not equal to one, ΔK_i is revised for all i, by

$$T = \min[qJ_i,\ (D_i - K_i)] \tag{7.18}$$

$$\Delta K_i = \max[T,\ (C_i - K_i)]. \tag{7.19}$$

Using the ΔK, we solve Δx and Δz from Equation 7.10 and then update x and z.

7.4 Algorithm for Quadratic Optimization

Since it is of practical importance, particular attention is given to the NLP that has the objective function and constraints described by quadratic forms [1,7]. This type of problem is referred to as quadratic optimization. The special case where the constraints are of linear forms is known as QP. Derivation of the sensitivity method is aimed at solving the NLP on the computer. An algorithm is generated for this purpose according to the result obtained in the previous section.

Quadratic optimization is involved in power systems [15] for maintaining a desirable voltage profile, maximizing power flow, and minimizing generation cost. These quantities are controlled by complex power generation which is usually bounded by two limits. The first two problems can be formulated with an objective function in a quadratic form of voltage while the last one is in a quadratic form of real power. Formulation of the first problem is given in the last of the illustrative problems.

As usual, we consider only minimization since maximization can be achieved by changing the sign of the objective function. Let the objective and constraint functions be expressed, respectively, by

$$f(x) = \frac{1}{2}x^{\mathrm{T}} Rx + a^{\mathrm{T}}x$$

and

$$f_i(x) = \frac{1}{2}x^{\mathrm{T}}H_i x + b_i^{\mathrm{T}}x \quad \text{for all } i. \tag{7.20}$$

R together with H are n-square and symmetrical matrices, and x, a together with b, are n-vectors. The quadratic functions are now characterized by the matrices and vectors. As defined before, the constraints are bounded by $C_i \le f_i(x) \le D_i$, for all $i = 1, 2, \ldots, m$. Among these i, the first p are equalities ($C_i = D_i$, for $i \le p$).

The matrix A and n-vector U in Equation 7.10 can be found by using

$$F_x^{\mathrm{T}} = [H_1x + b_1, H_2x + b_2, \ldots, H_mx + b_m],$$

$$S = R + \sum_{i=1}^{m} \lambda_i H_i, \quad w = a + \sum_{i=1}^{m} \lambda_i b_i,$$

and $U = Sx + w$.

Given below is an algorithm to be implemented in a computer program.

1. ***Input Data***
   (a) n, m, p, and ε (to replace zero, usually lies between 10^{-3} and 10^{-5}).
   (b) R, a, H_i, b_i, C_i, and D_i for all $i = 1, 2, \ldots, m$.

2. ***Initialization*** Set $x_i = 0$ and $z_i = 0$ ($\lambda_i = 0$) for all i or use any other preference.
3. ***Testing EKT Conditions***
   (a) Calculate K_i, U_i (the ith component of U) and then J_i from Equations 7.11 and 7.12 for all i.
   (b) A set of EKT conditions is reached if $|U_i| < \varepsilon$ and $|J_i| < \varepsilon$ for all i. Otherwise go to step 4.
4. ***Solving for u and v***
   (a) Solve u and v from Equation 7.15 by using LSMN [13].
   (b) Calculate N by Equation 7.16 and then go to step 5 if $N > 0$ and $f_x u < 0$. Go to Part (c) otherwise.
   (c) Update x by $x + u$, and $z + v$, and then go to step 3.
5. ***Determining ΔK***
   (a) Calculate q by Equation 7.17 and then find ΔK_i from Equations 7.18 and 7.19 for all i.
   (b) Solve Δx and Δz from Equation 7.10 by using LSMN.
   (c) Update x by $x + \Delta x$ and z by $z + \Delta z$, and then go to step 3.

In using the algorithm, one should discover several set of EKT conditions. This can be done by varying the initial values. Sometimes, intuitive judgment is helpful in deciding if the smallest one is the solution of the problem.

7.5 Illustrative Example (Barrier Method for Solving NLP)

As given by Equation 7.1 in the sensitivity method, the NLP is rewritten here as

$$\left.\begin{array}{l} \text{Minimize } f(x) \\ \text{Subject to } g(x) = 0 \\ \text{and } C \le h(x) \le D \end{array}\right\}, \tag{7.21}$$

where
x is an n-vector
$f(x)$ is a scalar function

The constraints $g(x)$ and $h(x)$ are, respectively, p- and m-vector functions. The bound vectors C and D are constant. All the functions are assumed to be twice differentiable. It is important to mention that m may be greater than n but p cannot. Any bound imposed on x may be considered as part of $h(x)$.

eastern Australia. The territory of NEM includes five states, New South Wales, South Australia, Queensland, Victoria, and Tasmania, and the Australian Capital area. The market operator is Australia Energy Market Operator (AEMO). AEMO is responsible for both power system operation and market operation of NEM. The National Electricity Market includes a power pool and a spot market. Generators, electricity retailers, and large customers trade electricity in the wholesale market through power pool. Retailers sell electricity to customers in electricity retail markets. AEMO is a *doing everything* operator on the NEM territory. It is responsible for generation dispatch, grid operation, system reliable and safety operation, market operation, and market settlement in the NEM region.

New Zealand reformed the power industry during the similar time as Australia. The wholesale electricity market, New Zealand Electricity Market (NZEM), was established in 1996. It has the world first nodal pricing-based market model. Transmission grid is owned and operated by the state-owned transmission company, Transpower. M-co, which is the former Electricity Market Company, manages the electricity market as a market regulator. Generators and retailers trade in the power pool and spot market, and electricity is traded and transmitted across the nation-wide transmission grid. Electricity price is determined for each node on a half-hour basis.

7.3.3 Nordic countries

The Nordic market was commenced in 1991 in Norway. One of the incentives to introducing power exchange in Norway was the lack of water in the hydro power reservoirs in 1991, which was a dry year. In Norway, 99% of electricity is generated by hydro power. Nord Pool was established in Oslo of Norway as a Power Exchange. Sweden, Finland, West Denmark, and East Demark joined the Nordic electricity market operated by Nord Pool. Denmark has two separate transmission systems in West Denmark and East Denmark, respectively. Power system restructuring in Nordic countries is similar. Generation companies are separated from national power grids. Generators and retailers/large customers trade electricity in the wholesale market. Electricity retail markets are open to industrial customers, commercial customers, and residential customers. Transmission grids are owned and operated by state transmission companies (TransCos). In Norway, Statnett is the TSO, which owns and operates the Norwegian transmission grid. Svenska Kraftnat (Svk) is the TSO of the Swedish transmission grid. In Finland, Fingrid is the TSO. The Danish transmission system is owned and operated by Energynet.dk.

$$\nabla g = \left(\frac{\partial g}{\partial x}\right)^{\mathrm{T}} = \left[\frac{\partial g_i}{\partial x_j}\right]^{\mathrm{T}} \quad (n \times p \text{ matrix}),$$

$$\nabla h = \left(\frac{\partial h}{\partial x}\right)^{\mathrm{T}} = \left[\frac{\partial h_i}{\partial x_j}\right]^{\mathrm{T}} \quad (n \times m \text{ matrix}),$$

$$\nabla^2 f = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right], \quad \nabla^2 g_k = \left[\frac{\partial^2 g_k}{\partial x_i \partial x_j}\right] \quad \text{and}$$

$$\nabla^2 h_k = \left[\frac{\partial^2 h_k}{\partial x_i \partial x_j}\right].$$

Enclosed by the brackets are the entry at the ith row and jth column. T denotes the transpose and the last three n-square matrices are Hessian matrices.

The K–T conditions can be obtained with respect to state and slack variables. For the state vector x, we have

$$\nabla L = \nabla f + (\nabla g)y + (\nabla h)(w - z) = 0. \tag{7.25}$$

Let S, R, W, and Z be the diagonal matrices that contain the elements of the vectors s, r, w, and z, respectively. Then the optimality conditions with respect to s and r are

$$-S^{-1}u + w = 0$$

and

$$-R^{-1}v + z = 0.$$

That is,

$$\left.\begin{aligned} Sw &= u \\ Rz &= v \end{aligned}\right\}. \tag{7.26}$$

The increment equations of Equation 7.26 are

$$\left.\begin{aligned} (S + \Delta S)(w + \Delta w) &= u \\ (R + \Delta R)(z + \Delta z) &= v \end{aligned}\right\}. \tag{7.27}$$

The penalty vectors u and v may alter each iteration but remain constant between iterations. By neglecting the terms $\Delta S \Delta w$ and $\Delta R \Delta z$, we obtain from Equation 7.27,

$$\left.\begin{aligned} \Delta w &= S^{-1}u - w - S^{-1}\Delta S w \\ \Delta z &= R^{-1}v - z - R^{-1}\Delta R z \end{aligned}\right\}. \tag{7.28}$$

Using the fact that $\Delta Sw = \overline{W}\Delta s$ and $\Delta Rz = Z\Delta r$, one may express Equation 7.28 as

$$\left.\begin{aligned} \Delta w &= S^{-1}u - w - S^{-1}\overline{W}\Delta s \\ \Delta z &= R^{-1}v - z - R^{-1}Z\Delta r \end{aligned}\right\}. \tag{7.29}$$

Increments of slack variables and state variables are linearly related if high orders of Δx are neglected. It follows from Equation 7.22 that

$$\left.\begin{aligned} \nabla h^{\mathrm{T}}\Delta x + \Delta s &= D - h - s = d_1 \\ \nabla h^{\mathrm{T}}\Delta x - \Delta r &= C - h + r = d_2 \end{aligned}\right\}. \tag{7.30}$$

Thus, we can write the relation between the increments as

$$\left.\begin{aligned} \Delta s &= d_1 - \nabla h^{\mathrm{T}}\Delta x \\ \Delta r &= \nabla h^{\mathrm{T}}\Delta x - d_2 \end{aligned}\right\}. \tag{7.31}$$

Substitutions of Equation 7.31 into 7.29 give

$$\left.\begin{aligned} \Delta w &= S^{-1}(u - \overline{W}d_1) - w + S^{-1}\overline{W}\nabla h^{\mathrm{T}}\Delta x \\ \Delta z &= R^{-1}(v + Zd_2) - z - R^{-1}Z\nabla h^{\mathrm{T}}\Delta x \end{aligned}\right\}. \tag{7.32}$$

The increment Equation 7.25, after all the variables are augmented, can be similarly determined. By neglecting the high orders of Δx and Δy, we have

$$(\nabla L)_{\mathrm{aug}} = \nabla L + H\Delta x + \nabla g\Delta y + \nabla h(\Delta w - \Delta z), \tag{7.33}$$

where

$$H = \nabla^2 f + \sum_{k=1}^{p} y_k \nabla^2 g_k + \sum_{k=1}^{m} (w_k - z_k)\nabla^2 h_k. \tag{7.34}$$

Substituting equals on Equation 7.32 into 7.33 and then setting Equation 7.33 equal to zero gives

$$(\nabla L)_{\mathrm{aug}} = A\Delta x + \nabla g\Delta y + b = 0, \tag{7.35}$$

where

$$A = H + \nabla h(S^{-1}\overline{W} + R^{-1}Z)\nabla h^{\mathrm{T}} \tag{7.36}$$

and

$$b = \nabla L + \nabla h(S^{-1}(u - \overline{W}d_1) - R^{-1}(v + Zd_2) - (w - z)). \tag{7.37}$$

The linearized equation of the equality constraint is

$$g + \nabla g^{\mathrm{T}} \Delta x = 0. \tag{7.38}$$

Combination of Equations 7.35 and 7.38 makes

$$\begin{bmatrix} A & \nabla g \\ \nabla g^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = - \begin{bmatrix} g \\ g \end{bmatrix}. \tag{7.39}$$

Being symmetrical, the $(n+p)$-square matrix on the left can be inverted by fast means even for large $n+p$. Computation time for Δx and Δy should not cause any problem in the process. Other increments can be readily found from Equations 7.31 and 7.29.

7.5.1.1 Analytical Forms

It is necessary to derive first the n-vector ∇f, $n \times p$ matrix ∇g, $n \times m$ matrix ∇h, and n-square symmetrical matrices $\nabla^2 f$, $\nabla^2 g_k$, and $\nabla^2 h_k$. Then form the n-vector ∇L and n-square matrix H in terms of the Lagrange multipliers according to Equations 7.25 and 7.34:

$$\nabla L = \nabla f + (\nabla g)y + \nabla h(w - z)$$

$$H = \nabla^2 f + \sum_{k=1}^{p} y_k \nabla^2 g_k + \sum_{k=1}^{m} (w_k - z_k) \nabla^2 h_k.$$

A wide class of NLP problems is expressible in the form of quadratic optimization. That is

$$f(x) = \frac{1}{2} x^{\mathrm{T}} Q x + a^{\mathrm{T}} x,$$

$$g_k(x) = \frac{1}{2} x^{\mathrm{T}} G_k x + B_k^{\mathrm{T}} x, \quad k = 1, 2, \ldots, p$$

$$h_k(x) = \frac{1}{2} x^{\mathrm{T}} h_k x + J_k^{\mathrm{T}} x, \quad k = 1, 2, \ldots, m,$$

where

Q, G_k, and H_k are symmetrical
a, B_k, and J_k are n-vectors

The vectors and matrices required by Equation 7.39 are

$$\nabla f = Qx + a$$
$$\nabla g = [G_1x + B_1, G_2x + B_2, \ldots, G_px + B_p]$$
$$\nabla h = [H_1x + J_1, H_2x + J_2, \ldots, H_mx + J_m]$$
$$\nabla^2 f = Q, \quad \nabla^2 g_k = G_k, \quad \text{and} \quad \nabla^2 h_k = H_k \text{ and}$$
$$H = Q + \sum_{k=1}^{p} y_k G_k + \sum_{k=1}^{m} (w_k - z_k) H_k.$$

7.5.1.2 Penalty Vectors

Each component of u or v may be chosen differently to achieve the discriminative penalty. However, one may choose the equipenalty scheme if there is no preference. That is,

$$u = v = \rho\mu e_m,$$

where $0 < \rho < 1$ and e_m is the m-vector with all elements equal to one. The penalty parameter μ is required to approach zero as the process approaches an optimum.

To meet such a requirement, we choose

$$\mu = \frac{1}{2m}(w^T s + z^T r) = \frac{1}{2m}(w_i s_i + z_i r_i). \tag{7.40}$$

Note that $\mu = 0$ at an optimum according to the K–T condition.

Start of Process. Initial values play an important part in the recursive process. Improper assumptions may cause divergence or convergence to a different solution (if one exists). It is known that the recursive process always converges to a solution if the initial values are close enough to it. If there is no preference, one may consider the following scheme.

State Vector.

1. Assume x to be an estimated solution.
2. Make x satisfy $\nabla f = 0$.
3. Set $x = 0$ if step 1 or 2 fails.

Slack Vectors.

Use $s = r = 1/2(D - C) > 0$.

Lagrange Vectors. Use $w = z = [1 + \|\nabla f\|]e_m$ and $y = 0$, where $\|\nabla f\|$ is the l_1-norm.

Penalty Parameter. $0.1 \leq \rho \leq 0.5$ may be used. A large number can retard the process and a small number can cause divergence.

Replacement of Zero. Two small numbers $\in_1$ and $\in_2$ are to be used to replace zero for computer implementation. They may be different and exist between 10^{-6} and 10^{-3}.

7.5.2 Computer Implementation

Using relevant equations in the earlier derivations:

1. ***Initialization*** Assume

$$x, s, r, w, \text{ and } z.$$

$$\rho, \in_1, \text{ and } \in_2.$$

Using current initial conditions,

2. ***Computation***

$$\mu = (1/2m)(w^{\mathrm{T}}s + z^{\mathrm{T}}r) \quad \text{and} \quad u = v = \rho\mu e_m. \tag{7.40}$$

$$d_1 = D - h - s \quad \text{and} \quad d_2 = C - h + r. \tag{7.30}$$

$$\nabla L = \nabla f + (\nabla g)y + (\nabla h)(w - z) = 0. \tag{7.25}$$

3. ***Computation***

$$A = H + \nabla h(S^{-1}\overline{W} + R^{-1}Z)\nabla h^{\mathrm{T}}. \tag{7.36}$$

$$b = \nabla L + \nabla h\big(S^{-1}(u - \overline{W}d_1) - R^{-1}(v + Zd_2) - (w - z)\big). \tag{7.37}$$

4. ***Increments***

$$\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} = -A^{-1}\begin{bmatrix}b\\ g\end{bmatrix} \tag{7.39}$$

$$\left.\begin{aligned}\Delta s &= d_1 - \nabla h^T \Delta x\\ \Delta r &= \nabla h^T \Delta x - d_2\end{aligned}\right\} \tag{7.31}$$

$$\left.\begin{aligned}\Delta w &= S^{-1}u - w - S^{-1}\Delta S w\\ \Delta z &= R^{-1}v - z - R^{-1}\Delta R z\end{aligned}\right\}. \tag{7.28}$$

5. ***Size of increment*** Determine two numbers according to

$$N_1 = \min\left[\frac{\Delta s_i}{s_i}, \frac{\Delta r_i}{r_i}, \quad i = 1, 2, \ldots, m\right]$$

$$N_2 = \min\left[\frac{\Delta w_i}{w_i}, \frac{\Delta z_i}{z_i}, \quad i = 1, 2, \ldots, m\right]$$

Then, set

$$\beta_1 = 1 \quad \text{if } N_1 \geq -1$$
$$= (-1/N_1) \quad \text{if } N_1 < -1$$
$$\beta_2 = 1 \quad \text{if } N_2 \geq -1$$
$$= (-1/N_2) \quad \text{if } N_2 < -1$$
$$\Delta x = \beta_1 \Delta x, \quad \Delta s = \beta_1 \Delta s, \quad \Delta r = \beta_1 \Delta r,$$
$$\Delta w = \beta_2 \Delta w, \quad \Delta z = \beta_2 \Delta z, \quad \text{and} \quad \Delta y = \beta_2 \Delta y.$$

6. ***Update***

$$\Delta x = \beta_1 \Delta x, \quad \Delta s = \beta_1 \Delta s, \quad \Delta r = \beta_1 \Delta r, \quad \text{and} \quad \Delta y = \beta_2 \Delta y.$$

Note that all the slack variables and Lagrange multipliers are non-negative due to the choice of β_1 and β_2.

7. ***Test for Termination***

Compute

$$\mu = \frac{1}{2m}(w^T s + z^T r) \tag{7.40}$$

$$\nabla L = \nabla f + (\nabla g)y + (\nabla h)(w - z) = 0. \tag{7.25}$$

Go to step 9 if both $\mu = \leq \epsilon_1$ and $\|\nabla L\| \leq \epsilon_2$ (l_1-norm) are satisfied. Otherwise, go to step 8.

8. ***Adjustment*** Go back to step 2 if both $N_1 \geq -0.995$ and $N_2 \geq -0.995$. Otherwise, go back to step 2 after having adjusted all the variables as follows.

$$x = x - 0.005\Delta x, \quad s = s - 0.005\Delta s, \quad r = r - 0.005\Delta r,$$
$$w = w - 0.005\Delta w, \quad z = z - 0.005\Delta z, \quad \text{and} \quad y = y - 0.005\Delta y.$$

9. Stop with solution: (x, y, s, r, w, and z).

7.6 Illustrative Examples

Example 7.6.1

Minimize $f = (x_1 - 2)^2 + 4$

Subject to

$x_1^2 + x_2^2 - 1 \leq 0, \quad x_1, x_2 \geq 0.$

Obtain the solution using the K–T condition.

SOLUTION

We can change the constraint to be two-sided as all the values of x_1, x_2 should be greater than zero. The constraint will be in the form $0 \leq x_1^2 + x_2^2 \leq 1$.

1. Form the Lagrange function

$$L(x, \lambda) = (x_1 - 2)^2 + 4 + \lambda(x_1^2 + x_2^2).$$

2. Search for the optimum candidates $L_x = 0$:

$$L_{x1} = 2(x_1 - 2) + 2\lambda x_1 = 0$$
$$L_{x2} = 2\lambda x_2 = 0.$$

Solving for the second equation for $L_{x2} = 0$, we have two possible solutions ($x_1 = 0$, $\lambda = 0$). But also using the K–T condition, which can be stated as follows,

$$\lambda = 0 \Rightarrow 0 < \left(x_1^2 + x_2^2 < 1\right)$$
$$\lambda \geq 0 \Rightarrow \left(x_1^2 + x_2^2\right) = 1$$
$$\lambda \leq 0 \Rightarrow \left(x_1^2 + x_2^2\right) = 0$$

a. At $\lambda = 0$, solving for $L_{x1} = 0$ results in $x_1 = 2$, which means that the main constraint, $0 \leq x_1^2 + x_2^2 \leq 1$ violates. Then there is no solution at $\lambda = 0$.

b. $\lambda \geq \mathbf{0}$, $x_2 = 0$, based on the K–T condition $\left(x_2^2 + x_2^2\right) = 1$, means that $x_1 = \pm 1$,

$$f(-1,0) = (-1 - 2)^2 + 4 = 13, \quad \lambda = -3 \text{ (out of range)}$$
$$f(1,0) = (1 - 2)^2 + 4 = 5, \quad \lambda = 1 \text{ (in range)}$$

c. $\lambda \leq 0$, $x_2 = 0$, based on the K–T condition $\left(x_1^2 + x_2^2\right) = 0$, means that $x_1 = 0$,

$$f(0,0) = (0 - 2)^2 + 4 = 8.$$
$$\text{Then } f_{\min} = f(1,0) = 5.$$

Example 7.6.2

Perform 1 iteration in solving the following problem:

Minimize $F = 0.25x_1^2 + x_2^2$

s.t. $1 \leq x_1 - x_2 \leq 7$

SOLUTION

$$L(x,\lambda) = f(x) + \lambda F(x) = 0.25x_1^2 + x_2^2 + \lambda(x_1 - x_2)$$

$$U(x,z) = f_x^T(x) + zF_x^T(x) = 0$$

$$f_x = x_1 - x_2 \Rightarrow f_{xx} = [1, -1]$$

$$F_x = [0.5x_1 \quad 2x_2]$$

$$S = U_x(x,z) = f_{xx} + \sum_{i=1}^{3} z_i f_{ixx}(x) = \begin{bmatrix} \lambda^T(1 - x_2) + 1 & \lambda^T(x_1 - 1) + 1 \\ \lambda^T(1 - x_2) - 1 & \lambda^T(x_1 - 1) - 1 \end{bmatrix}$$

$$U_z(x,z) = F_x^T(x) = \begin{bmatrix} 0.5x_1 \\ 2x_2 \end{bmatrix}$$

Note $A_y = b$ where $A = \begin{bmatrix} S & F_x^T \\ F_x & 0 \end{bmatrix}$, $y = \begin{bmatrix} \Delta x \\ \Delta z \end{bmatrix}$, and $b = \begin{bmatrix} -u \\ \Delta k \end{bmatrix}$

Using initial solution as $x = [1, 0]$, $\lambda = -2$, and $k = 3$, we get

$$S = \begin{bmatrix} 2x_2 - 1 & 3 - 2x_1 \\ 2x_2 - 3 & 1 - 2x_1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -3 & -1 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}$$

$$F_x = [0.5 \quad 0], \quad U = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

If $A_y = b$, then $y = A^{-1}b$

$$\therefore y = \begin{bmatrix} S^{-1} & (F_x^T)^{-1} \\ F_x^{-1} & 0 \end{bmatrix} \begin{bmatrix} -U \\ \Delta K \end{bmatrix} = \begin{bmatrix} S^{-1}(-u) + (F_x^T)^{-1}(\Delta k) \\ F_x^{-1}(-u) + 0 \end{bmatrix}$$

$$S^{-1}(-u) = \begin{bmatrix} -0.25 & -0.25 \\ 0.75 & -0.25 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.25 \\ -0.25 \end{bmatrix}$$

$$(F_x^T)^{-1}(\Delta k) = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$(F_x)^{-1}(-u) = [2 \quad 0] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

$$y = \begin{bmatrix} \Delta x \\ \Delta z \end{bmatrix} = \begin{bmatrix} [5.25 - 0.25]^T \\ 0 \end{bmatrix}, \quad \Delta x = \begin{bmatrix} 5.25 \\ -0.25 \end{bmatrix}, \quad \Delta z = \Delta\lambda^T = 0$$

using Newton Method: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{new} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{old} + \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 5.25 \\ -0.25 \end{bmatrix} = \begin{bmatrix} 6.25 \\ -0.25 \end{bmatrix}$$

$$\lambda^{new} = \lambda^{old} + \Delta\lambda = -2 + 0 = -2$$

using $x_1 - x_2 \Rightarrow 6.25 - (-0.25) = 6.25$

$f_{min}(6.25, -0.25) = 0.25(6.25)^2 + (-0.25)^2 = 9.828$ after 1 iteration.

Example 7.6.3

Minimize $f(x_1,x_2) = (x_1 - 1)^2 + x_2 - 2$

s.t. $h(x_1,x_2) = x_2 - x_1 - 1 = 0$

$g(x_1,x_2) = x_1 - x_2 - 2 \leq 0$

Solution

Formulate the Lagrangian as

$$\begin{aligned} L &= f(x_1,x_2) + \lambda_1 h(x_1,x_2) + \lambda_2 g(x_1,x_2) \\ &= (x_1 - 1)^2 + x_2 - 2 + \lambda_1(x_2 - x_1 - 1) + \lambda_2(x_1 - x_2 - 2) \end{aligned}$$

Necessary condition

$$\frac{\partial L}{\partial x_1} = 2(x_1 - 1) - \lambda_1 + \lambda_2 = 0 \Rightarrow x_1 = \frac{\lambda_1 - \lambda_2}{2} + 1 \quad (1)$$

$$\frac{\partial L}{\partial x_2} = 1 + \lambda_1 - \lambda_2 = 0 \Rightarrow \lambda_1 = \lambda_2 - 1 \quad (2)$$

From Equations 1 and 2, we get

$$\therefore x_1 = \frac{\lambda_2 - 1 - \lambda_2}{2} + 1 = 0.5 \quad (3)$$

Substitute Equation 3 into $h(x_1,x_2)$, we obtain $x_2 = 1.5$. Hence $(x_1,x_2) = (0.5, 1.5)$
Sufficiency condition

$$\frac{\partial^2 L}{\partial x_1^2} = 2 \quad \frac{\partial^2 L}{\partial x_2 \partial x_1} = 0$$

$$\frac{\partial L}{\partial x_1 \partial x_2} = 0 \quad \frac{\partial^2 L}{\partial x_2^2} = 0$$

$H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is positive definite.

$(x_1,x_2) = (0.5,1.5)$ is the minimum point. $f(0.5,1.5) = (0.5 - 1)^2 + 1.5 - 2 = -0.25$ is minimum value of $f(x_1,x_2)$.

Example 7.6.4

Use the barrier algorithm to minimize the function: $f(x) = x_1 x_2$
Subject to $1 \le x_1 - x_2 \le 2$.

Solution

With the number of variables $n = 2$ and number of constraints $m = 1$,

$$h(x) = x_1 - x_2, \quad D = 2, \quad C = 1.$$

Use the form:

$$f(x) = \frac{1}{2} x^T Q x + a^T x = x_1 x_2.$$

Then

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad a = 0$$

$$\text{Subject to } 1 \le x_1 - x_2 \le 2, \quad \text{then } h(x) = x_1 - x_2$$

$$h_k(x) = \frac{1}{2} x^T H_k x + J_k^T x, \quad \text{then } k = 1,\ H_1 = 0$$

$$J_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$H = Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\nabla f = Qx + a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

$$\nabla h = H_1 x + J_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \nabla^2 f = Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\nabla L = \nabla f + (w - z)\nabla h = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + (w - z) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x_2 + w - z \\ x_1 - w + z \end{bmatrix}.$$

First iteration

1. Initialization

$$s = r = \frac{1}{2}(D - C) = \frac{1}{2}(2 - 1) = 0.5.$$

Set $x_1 = x_2 = 0$ to satisfy $\nabla f = 0$,

$$w = z = [1 + \|\nabla f\|]e_m,$$

where e_m is an m-vector with element 1

$$w = z = (1+0)1 = 1$$

Take $\rho = 0.5$, $\varepsilon_1 = 10^{-2}$, and $\varepsilon_2 = 10^{-5}$.

2. Computation

(a) $$\mu = \frac{1}{2m}(w^T s + z^T r) + \frac{1}{2}(0.5 + 0.5) = 0.5$$
$$u = v = \rho\mu e_m = 0.25.$$

(b) $d_1 = D - h - s$, $h = x_1 - x_2 = 0 - 0 = 0$ then
$$d_1 = 2 - 0 - 0.5 = 1.5$$
$$d_2 = C - h + r = 1 - 0 + 0.5 = 1.5.$$

(c) $$\nabla L = \begin{bmatrix} x_2 + w - z \\ x_1 - w + z \end{bmatrix} = \begin{bmatrix} 0 - 1 + 1 \\ 0 + 1 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

3. Computation

(a) $$A = H + \nabla h(s^{-1}w + r^{-1}z)\nabla h^T$$
$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (2+2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}[1 \quad -1] = \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}.$$

(b) $$b = \nabla L + \nabla h\{s^{-1}(u - wd_1) - r^{-1}(v + zd_2) - (w - z)\}$$
$$b = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\{2(0.25 - 1 \times 1.5) - 2(0.25 + 1 \times 1.5) - 0\}$$
$$= \begin{bmatrix} -6 \\ 6 \end{bmatrix}.$$

4. Increments

$$\begin{bmatrix} \nabla x \\ \nabla y \end{bmatrix} = -A^{-1}\begin{bmatrix} b \\ g \end{bmatrix},$$

$$\Delta x = -A^{-1}b = -\begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}^{-1}\begin{bmatrix} -6 \\ 6 \end{bmatrix} = 0.8571\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Delta s = d_1 - \nabla h^T \Delta x \nabla h = 1.5 - 0.8571[1 \quad -1]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -0.2143$$

$$\Delta r = \nabla h^T \Delta x - d_2 = 0.2143$$

$$\Delta w = s^{-1}(u - w\Delta s) - w = 0.1286$$

$$\Delta z = r^{-1}(v - z\Delta r) - z = -0.7286.$$

5. Size of increment

$$N_1 = \min\left\{\frac{\Delta s}{s}, \frac{\Delta r}{r}\right\} = \min\left\{\frac{-0.2143}{0.5}, \frac{0.2143}{0.5}\right\} = -0.42856$$

$$N_2 = \min\left\{\frac{\Delta w}{w}, \frac{\Delta z}{z}\right\} = \min\left\{\frac{0.1286}{1}, \frac{-0.7286}{1}\right\} = -0.7286$$

$$N_1 > -1 \Rightarrow \beta_1 = 1$$

$$N_2 > -1 \Rightarrow \beta_2 = 1.$$

Then there is no change in the previous calculated increments.

6. Update

$$s = s + \Delta s = 0.5 - 0.2143 = 0.2857$$

$$x = x + \Delta x = \begin{bmatrix} 0.85714 \\ -0.85714 \end{bmatrix}$$

$$r = r + \Delta r = 0.5 + 0.2143 = 0.7143$$

$$w = w + \Delta w = 1 + 0.1286 = 1.1286$$

$$z = z + \Delta z = 1 - 0.7286 = 0.2714.$$

7. Test of termination

$$\mu = \frac{1}{2m}(w^{\mathrm{T}}s + z^{\mathrm{T}}r) = \frac{1}{2 \times 1}(1.1286 \times 0.2857 + 0.2714 \times 0.7143)$$
$$= 0.2582 > \varepsilon_1$$

$$\nabla L = \begin{bmatrix} x_2 + w - z \\ x_1 - w + z \end{bmatrix} = \begin{bmatrix} -0.85714 + 1.1286 - 0.2714 \\ 0.85714 - 1.1286 + 0.2714 \end{bmatrix}$$
$$= -0.331 \times 10^{-15} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\|\nabla L\| = 4.7103 \times 10^{-16} > \varepsilon_2.$$

8. Adjustment

$N_1 = -0.4286 > -0.995$ and $N_2 = -0.7286 > -0.995.$

Then go to step 2.

Second iteration

Using current initial conditions,

2. Computation

(a) $$\mu = \frac{1}{2m}(w^{\mathrm{T}}s + z^{\mathrm{T}}r) = 0.2582$$
$$u = v = \rho\mu e_{\mathrm{m}} = 0.1807.$$

(b) $d_1 = D - h - s, \quad h = x_1 - x_2 = 0.85714 + 0.85714 = 1.71428.$

Then

$$d_1 = 1.0e-04$$

$$d_2 = C - h + r = 1 - 1.7142 + 0.7143 = 1.0e-04.$$

(c)
$$\nabla L = \begin{bmatrix} x_2 + w - z \\ x_1 - w + z \end{bmatrix} = \begin{bmatrix} -0.85714 + 1.1286 - 0.2714 \\ 0.85714 - 1.1286 + 0.2714 \end{bmatrix}$$
$$= -0.331 \times 10^{-15} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

3. Computation

(a)
$$A = H + \nabla h(s^{-1}w + r^{-1}z)\nabla h^{\mathrm{T}}$$
$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \left(\frac{1.1286}{0.2857} + \frac{0.2714}{0.7143}\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}[1 \quad -1]$$
$$= \begin{bmatrix} 4.3302 & -3.3302 \\ -3.3302 & 4.3302 \end{bmatrix}.$$

(b)
$$b = \nabla L + \nabla h\{s^{-1}(u - wd_1) - r^{-1}(v + zd_2) - (w - z)\}$$
$$= \begin{bmatrix} -0.4780 \\ 0.4780 \end{bmatrix}.$$

4. Increments

$$\begin{bmatrix} \nabla x \\ \nabla y \end{bmatrix} = -A^{-1}\begin{bmatrix} b \\ g \end{bmatrix}$$

$$\Delta x = -A^{-1}b = -\begin{bmatrix} 4.3302 & -3.3302 \\ -3.3302 & 4.3302 \end{bmatrix}^{-1}\begin{bmatrix} -0.478 \\ 0.478 \end{bmatrix} = 0.0624\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Delta s = d_1 - \nabla h^{\mathrm{T}}\Delta x\nabla h = -0.1247$$

$$\Delta r = \nabla h^{\mathrm{T}}\Delta x - d_2 = 0.1247$$

$$\Delta w = s^{-1}(u - w\Delta s) - w = -0.0035$$

$$\Delta z = r^{-1}(v - z\Delta r) - z = -0.0658.$$

5. Size of increment

$$N_1 = \min\left\{\frac{\Delta s}{s}, \frac{\Delta r}{r}\right\} = \min\left\{\frac{-0.1247}{0.2857}, \frac{0.1247}{0.7143}\right\} = -0.4365$$

$$N_2 = \min\left\{\frac{\Delta w}{w}, \frac{\Delta z}{z}\right\} = \min\left\{\frac{-0.0035}{1.1286}, \frac{-0.0658}{0.2714}\right\} = -0.2424$$

$$N_1 > -1 \Rightarrow \beta_1 = 1$$

$$N_2 > -1 \Rightarrow \beta_2 = 1.$$

Then there is no change in the previous calculated increments.

6. Update

$$s = s + \Delta s = 0.2857 - 0.1247 + 0.161$$

$$x = x + \Delta x = \begin{bmatrix} 0.8571 \\ -0.8571 \end{bmatrix} + \begin{bmatrix} 0.0624 \\ -0.0624 \end{bmatrix} = \begin{bmatrix} 0.9195 \\ -0.9195 \end{bmatrix}$$

$$r = r + \Delta r = 0.7143 + 0.1247 = 0.8390$$

$$w = w + \Delta w = 1.1286 - 0.0035 = 1.1251$$

$$z = z + \Delta z = 0.2714 - 0.0658 = 0.2056.$$

7. Test of termination

$$\mu = \frac{1}{2m}(w^{\mathrm{T}}s + z^{\mathrm{T}}r) = \frac{1}{2 \times 1}(1.1251 \times 0.161 + 0.2056 \times 0.8390)$$
$$= 0.1768 > \varepsilon_1$$

$$\nabla L = \begin{bmatrix} x_2 + w - z \\ x_1 - w + z \end{bmatrix} = \begin{bmatrix} -0.9195 + 1.1251 - 0.2056 \\ 0.9195 - 1.1251 + 0.2056 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} < \varepsilon_2.$$

8. Adjustment

$$N_1 = -0.4365 > -0.995 \quad \text{and} \quad N_2 = -0.2424 > -0.995.$$

Then go to step 2.

Repeating the previous step, we get the solution that satisfies the conditions $\varepsilon_1 = 10^{-2}$ and $\varepsilon_2 = 10^{-5}$ after 10 iterations. Table 7.1 tabulates the results of these iterations.

The solution is $x_1 = 0.995$ and $x_2 = -0.995$.

TABLE 7.1

Results of All Iterations for Example 7.6.2

Iteration Number	x_1	x_2	r	s	w	z	μ	$\|\Delta A\|$
1	0.8571	−0.8571	0.7143	0.2857	1.1286	0.2714	0.2585	4.71 E-16
2	0.9195	−0.91915	0.8390	0.1610	1.1251	0.2056	0.1768	0.0
3	0.9416	−0.9416	0.8833	0.1167	1.0782	0.1367	0.1233	1.57 E-16
4	0.9585	−0.9585	0.9170	0.0830	1.0510	0.0925	0.0860	4.79 E-15
5	0.9707	−0.9707	0.9414	0.0586	1.0339	0.0632	0.0601	0.0
6	0.9794	−0.9794	0.9587	0.0413	1.0228	0.0435	0.0420	4.71 E-16
7	0.9855	−0.9855	0.9710	0.0290	1.0156	0.0301	0.0293	1.57 E-16
8	0.9898	−0.9898	0.9796	0.0204	1.0107	0.0209	0.0205	1.57 E-16
9	0.9929	−0.9929	0.9857	0.0143	1.0074	0.0146	0.0144	3.14 E-16
10	0.995	−0.995	0.9900	0.0100	1.0051	0.0102	0.0100	1.884 E-15

7.7 Conclusion

This chapter dealt with optimization techniques that fit most nonlinear engineering applications. It presented several algorithms in NLP that are programmable by scientists and engineers using well developed numerical methods such as in [14]. NLP aims at solving optimization problems involving nonlinear objective and constrained functions [3–9,16]. In Sections 7.1 and 7.2, classification of NLP problems was presented. The classification included quadratic, convex, and separable programming. The sensitivity method for solving NLP variables was presented in Section 7.3. Also, a practical procedure for solving the problem was demonstrated with an alternative expression for the extended K–T condition to provide feasibility and optimality. In Section 7.4 an algorithm for solving the quadratic optimization problem was presented in the form of sequential steps. A technique based on the barrier method for solving NLP problems was presented in Section 7.5, where the recursive process was developed.

7.8 Problem Set

PROBLEM 7.8.1

Solve the following as a separate convex programming problem.

Minimize $Z = (x_1 - 2)^2 + 4(x_2 - 6)^2$

Subject to

$6x_1 + 3(x_2 + 1)^2 \leq 12$

$x_1, x_2 \geq 0$

PROBLEM 7.8.2

Consider the problem

Maximize $Z = 6x_1 + 3x_2 - 4x_1x_2 - 2x_1^2 - 3x_2^2$

Subject to

$x_1 + x_2 \leq 1$

$2x_1 + 3x_2 \leq 4$

$x_1, x_2 \geq 0.$

Show that Z is strictly concave and then solve the problem using the QP algorithm.

PROBLEM 7.8.3

Consider the problem

Minimize $Z = x_1^2 + x_2^2$

Subject to

$2x_1 + x_2 \leq 2$

$-x_1 + 1 \leq 0.$

1. Find the optimal solution to this problem.
2. Formulate a suitable function with initial penalty parameter $\mu = 1$.
3. Starting from the point (2, 6), solve the resulting problem by a suitable unconstrained optimization technique.
4. Replace the penalty parameter μ by 10. Starting from the point obtained in 3, solve the resulting problem.

PROBLEM 7.8.4

Minimize $f = (x_1 + 1)(x_2 - 2)$ over the region $0 \leq x_1 \leq 2$, $0 \leq x_2 \leq 1$ by writing the K–T conditions and obtaining the saddle point.

PROBLEM 7.8.5

Minimize $Z = 2x_1 - x_1^2 + x_2$

Subject to

$2x_1 + 3x_2 \leq 6$

$x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0.$

PROBLEM 7.8.6

Minimize $Z = (x_1 - 6)^2 + (x_2 - 8)^2$

Subject to

$x_1^2 - x_2^2 \leq 0.$

Using the auxiliary function $(x_1 - 6)^2 + (x_2 - 8)^2 + \mu \max(x_1^2 - x_2, 0)$, and adopting the cyclic coordinate method, solve the above problem starting from $x_1 = (0, -4)^t$ under the following strategies for modifying μ.

1. Starting from $\mathbf{x}_1$, solve the penalty problem for $\mu_1 = 0.1$ resulting in $\mathbf{x}_2$. Then start from $\mathbf{x}_2$, and solve the problem with $\mu_2 = 100$.
2. Starting from the unconstrained optimal point (6, 8), solve the penalty problem for $\mu_2 = 100$. (This is the limiting case of Part 1 for $\mu_1 = 0$.)
3. Starting from $\mathbf{x}_1$, solve the penalty problem $\mu_1 = 100.0$.

4. Which of the above strategies would you recommend, and why? Also, in each of the above cases, derive an estimate for the Lagrangian multiplier associated with the single constraint.

PROBLEM 7.8.7

Maximize $f(x) = x_1^2 - 2x_2^2$

Subject to

$x_1 - x_2 = 2$

$x_2 \leq x_1^2 \leq x_2 + 8.$

PROBLEM 7.8.8

Maximize $f(x) = 3x_1x_2 - x_1^2 - x_2^2$

Subject to

$x_1 \leq 0$ and $0 \leq x_2 \leq 1.$

PROBLEM 7.8.9

Minimize $x_1^2 + x_2^2$

s.t. $x_1 - x_2^2 - 4 \geq 0$

$x_1 - 10 \leq 0$

PROBLEM 7.8.10

Use KT conditions to solve

Minimize $f(x_1,x_2) = (x_1 - 1)^2 + x_2 - 2$

s.t. $h(x_1,x_2) = x_2 - x_1 - 1 = 0$

$f(x_1,x_2) = x_1 - x_2 - 2 \leq 0$

PROBLEM 7.8.11

Apply the NLP sensitivity method to solve

Minimize $f(x_1,x_2) = 0.5x_1^2 + 2x_2^2$

s.t. $2 \leq 2x_1 - 2x_2 \leq 4$

PROBLEM 7.8.12

Minimize $f(x_1,x_2) = x_1^2 + x_1x_2 + 2x_2^2 - 6x_1 - 14x_2$

s.t. $3 \leq x_2 + x_1 \leq 6$

$2 \leq -x_1 + 2x_2 \leq 3$

PROBLEM 7.8.13

Minimize $f(x_1,x_2) = 0.25x_1^2 + x_2^2$

s.t. $1 \leq x_1 - x_2 \leq 2$

PROBLEM 7.8.14

Consider the NLP problem:

Minimize $x_1^2 + x_2^2 + 2x_1x_2 + 2x_1$

s.t. $1 \leq x_1 + x_2 \leq 2$

Solve by

a. Sensitivity/Newtonian method
b. Quadratic interior point method

(Perform at least one iteration for each method)

PROBLEM 7.8.15

Apply K–T conditions to solve the NLP problem given in Problem 7.8.14.

PROBLEM 7.8.16

Maximize $f(x) = -(2x_1 - 5)^2 - (2x_2 - 1)^2$

Subject to

$x_1 + 2x_2 \leq 2$

$x_1, x_2 \geq 0$

PROBLEM 7.8.17

Consider the problem defined as Maximize $f(\boldsymbol{x})$ Subject to $h(\boldsymbol{x}) \leq 0$. If the Lagrangian is formulated as $L(\boldsymbol{x}, \lambda, s) = f(\boldsymbol{x}) + \lambda[h(\boldsymbol{x}) + s^2]$, how would this change affect the K–T conditions?

PROBLEM 7.8.18

Consider the following generalized nonlinear problem:

$Max(x)$

Subject to

$g_i(\boldsymbol{x}) \leq 0, \quad i = 1, 2, \ldots, r$

$g_i(\boldsymbol{x}) \geq 0, \quad i = r+1, \ldots, p$

$g_i(\boldsymbol{x}) \leq 0, \quad i = p+1, 2, \ldots, m$

a. Convert the inequality constraints to equality and construct and appropriate Lagrangian

$$L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{s}) = f(\boldsymbol{x}) - \sum_{i=1}^{r} \lambda_i \left[g_i(\boldsymbol{x}) + s^2 \right] - \sum_{i=r+1}^{p} \lambda_i \left[g_i(\boldsymbol{x}) - s^2 \right] - \sum_{i=p+1}^{m} \lambda_i g_i(\boldsymbol{x})$$

b. Develop the K–T conditions to the following Lagrange function

PROBLEM 7.8.19

Determine extreme points for the functions give by

a. $f(\boldsymbol{x}) = x_1^3 + 2x_2^3 - 4x_1x_2$
b. $f(\boldsymbol{x}) = 2x_1^2 + 3x_2^2 - x_3^2 + 4(x_1 + x_3) + 3x_2x_3$

PROBLEM 7.8.20

a. Minimize $(x_1 - 2)^4 + (x_1 - 2x_2)^2$
   s.t. $x_1^2 + x_2 = 0$
   using the Barrier algorithm
b. Using the quadratic interior point (QUIP) method
   Maximize $P = -2x_1 - 6x_2 + x_1^2 + x_2^2$
   s.t. $x_1 + 2x_2 \leq 5$
   $x_1 + x_2 \leq 3$ for all positive **x**.

PROBLEM 7.8.21

Consider the quadratic problem stated as
Minimize $0.5 \times {}^{\mathrm{T}}Hx + c^{\mathrm{T}}x$: $Ax = b$
Where $A_{m \times n}$ is of rank m and H is positive definite:

a. Show that the matrix $\begin{bmatrix} H & A^{\mathrm{T}} \\ A & 0 \end{bmatrix}$ is nonsingular.
b. Show that the linear equation of the KKT system of the QP yields a unique solution
c. Assume that H is positive definite and hence nonsingular, derive an explicit or closed form expression for the optimal solution to the QP problem.

PROBLEM 7.8.22

a. Determine whether or not the quadratic form

$$Q(x_1,x_2) = 7x_1 + 3x_2 - 4x_1x_2 - 2x_1^2 - 3x_2^2 \text{ is negative-definite.}$$

b. Determine if the quadratic form

$$Q(x_1,x_2,x_3) = 2x_1^2 + 5x_2^2 + 3x_3^2 + 4x_1x_2 + 6x_2x_3 \text{ is positive-definite.}$$

References

1. Hamdy, A. T. *Operations Research: An Introduction*, 8th edn., Prentice-Hall, New Jersey, 2006.
2. Himmelblau, D. M. *Applied Nonlinear Programming*, McGraw-Hill, New York, 1972.
3. Luenberger, D. G. *Introduction to Linear and Nonlinear Programming*, Addison-Wesley, Reading, MA, 1973.
4. Mangasarian, O. L. Nonlinear programming problems with stochastic objective functions, *Management Science*, 10, 1964, 353–359.
5. McCormick, G. P. Penalty function versus non penalty function methods for constrained nonlinear programming problems, *Mathematical Programming*, 1, 1971, 217–238.
6. McMillan, C., Jr. *Mathematical Programming*, Wiley, New York, 1970.
7. Murtagh, B. A. and Sargent, R. W. H. Computational experience with quadratically convergent minimization methods, *Computer Journal*, 13, 1970, 185–194.
8. Pierre, D. A. *Optimization Theory with Applications*, Wiley, New York, 1969.
9. Powell, M. J. D. A method for nonlinear constraints in minimization problems, in: *Optimization*, R. Fletcher (Ed.), Academic Press, London, New York, 1969.
10. Wilde, D. J. *Optimum Seeking Methods*, Prentice-Hall, Englewood Cliffs, NJ, 1964.
11. Wilde, D. J. and Beightler, C. S. *Foundations of Optimization*, Prentice-Hall, Englewood Cliffs, NJ, 1967.
12. Zangwill, W. I. *Nonlinear Programming: A Unified Approach*, Prentice-Hall, Englewood Cliffs, NJ, 1969.
13. Zoutendijk, G. Nonlinear programming, computational methods, in: *Integer and Nonlinear Programming*, J. Abadie (Ed.), North-Holland, Amsterdam, 1970, pp. 37–86.
14. Forsythe, G. E., Malcolm, M. A., and Moler, C.B. *Computer Methods for Mathematical Computations*, Prentice-Hall, Englewood Cliffs, NJ, 1977.
15. Momoh, J. A., Dias, L. G., Guo, S. X., and Adapa, R. A. Economic operation and planning of multi-area interconnected power system, *IEEE Transactions on Power Systems*, 10, 1995, 1044–1051.
16. Vanderbei, R. J. and Shanno, D. F. An interior point algorithm for nonconvex nonlinear programming, Research Report SOR-97-21, Statistics and Operations Research, Princeton University, Princeton, NJ.

8

Dynamic Programming

8.1 Introduction

Dynamic programming (DP) is an optimization approach that transforms a complex problem into a sequence of simpler problems [3–6,9,12–14]; its essential characteristic is the multistage nature of the optimization procedure. More so than the optimization techniques described previously, DP provides a general framework for analyzing many problem types. Within this framework a variety of optimization techniques can be employed to solve particular aspects of a more general formulation. Usually creativity is required before we can recognize that a particular problem can be cast effectively as a dynamic program, and often subtle insights are necessary to restructure the formulation so that it can be solved effectively.

The DP method was developed in the 1950s through the work of Richard Bellman [1,2] who is still the doyen of research workers in this field. The essential feature of the method is that a multivariable optimization problem is decomposed into a series of stages, optimization being done at each stage with respect to one variable only. Bellman [1] gave it the rather undescriptive name of DP. A more significant name would be recursive optimization.

Both discrete and continuous problems can be amenable to this method and deterministic as well as stochastic models can be handled by it. The complexities increase tremendously with the number of constraints. A single-constraint problem is relatively simple, but even more than two constraints can be formidable.

The DP technique, when applicable, represents or decomposes a multistage decision problem as a sequence of single-stage decision problems. Thus an N-variable problem is represented as a sequence of N single-variable problems that are solved successively. In most of the cases, these N subproblems are easier to solve than the original problem. The decomposition to N subproblems is done in such a manner that the optimal solution of the original N-variable problem can be obtained from the optimal solutions of the N one-dimensional problems. It is important to note that the particular optimization technique used for the optimization of the N-single-variable problems is irrelevant. It may range from a simple enumeration process to a calculus or a nonlinear programming technique.

Multistage decision problems can also be solved by the direct application of classical optimization techniques. However, this requires the number of variables to be small, the functions involved to be continuous and continuously differentiable, and the optimum points not to lie at the boundary points.

Furthermore, the problem has to be relatively simple so that the set of resultant equations can be solved either analytically or numerically. The nonlinear programming techniques can be used to solve slightly more complicated multistage decision problems. But their application requires the variables to be continuous and for there to be prior knowledge about the region of the global minimum or maximum. In all these cases, the introduction of stochastic variability makes the problem extremely complex and the problem unsolvable except by using some sort of an approximation-like chance constrained optimization.

DP, on the other hand, can deal with discrete variables, and nonconvex, noncontinuous, and nondifferentiable functions [4,7–11]. In general, it can also take into account the stochastic variability by a simple modification of the deterministic procedure. The DP technique suffers from a major drawback, known as the curse of dimensionality. However, in spite of this disadvantage, it is very suitable for the solution of a wide range of complex problems in several areas of decision making.

Optimization overtime in a single or multiple decision process is generally formulated as DP. It involves a large number of variables over different stages. DP is a procedure designed to improve the computation efficiency for solving a set of large-scale problems in time that can be decomposed into smaller and have computationally simple problems. DP solves the optimization in stages with each stage involving exactly one optimization variable. The computations in each stage are linked by the focus called recursive computation in a manner that is feasible in optimization solution. This will be obtained for the entire problem. When the last stage reached this there is an advantage for each stage being optimized. On the other hand, the advantage loss is in the complexity of its solution for a large system, the so-called curse of dimensionality. With this in mind DP, has limited applications and there are many recent works to advance this technology for possible applications in power systems management. These include candidates such as dynamic programming (ADP), artificial neural networks (ANN), and evolutionary programming in the area of genetic algorithms (GA), which are observed in later chapters.

8.2 Formulation of a Multistage Decision Process

8.2.1 Representation of a Multistage Decision Process

Any decision process is characterized by certain input parameters, X (or data), certain decision variables (U), and certain output parameters (T)

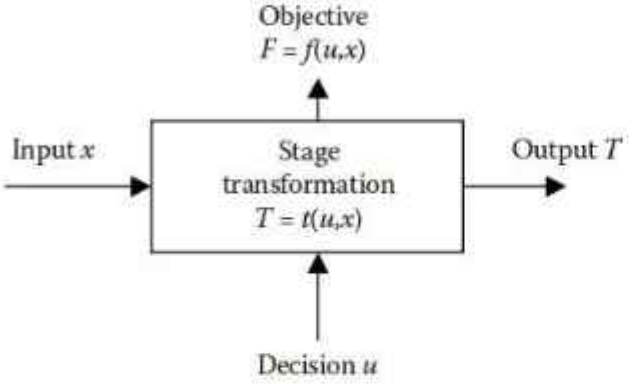


FIGURE 8.1
Single-stage decision problem.

representing the outcome obtained as a result of making the decision. The input parameters are called input stage variables, and the output parameters are called output stage variables. Finally there is a return or objective function F, which measures the effectiveness of the decisions for any physical system, represented as a single-stage decision process (shown in Figure 8.1). The output of this single stage is shown in Equations 8.1 and 8.2.

$$x_i = t_i(x_{i+1},u_i) \quad \forall\, i \in \{1,n\} \tag{8.1}$$

$$F_i = f_i(x_{i+1},u_i) \quad \forall\, i \in \{1,n\}, \tag{8.2}$$

where u_i denotes the vector of decision variables at stage i. The objective of a multistage decision process is to find $u_1, u_2, \ldots, u_n$ so as to optimize some function of the individual stage returns, say, $F(f_1, f_2, \ldots, f_n)$ and satisfy Equations 8.1 and 8.2. The nature of the n-stage return function f determines whether a given multistage problem can be solved as a decomposition technique; it requires the separability and monotonicity of the objective function.

In order to have separability of the objective function, we must be able to represent the objective function as the composition of the individual stage returns. This requirement is satisfied for additive objective functions:

$$F = \sum_{i=1}^{n} f_i(u_i,x_{i+1}), \tag{8.3}$$

where u_i are real, and for multiplicative objective functions:

$$F = \prod_{i=1}^{n} f_i(x_{i+1},u_i). \tag{8.4}$$

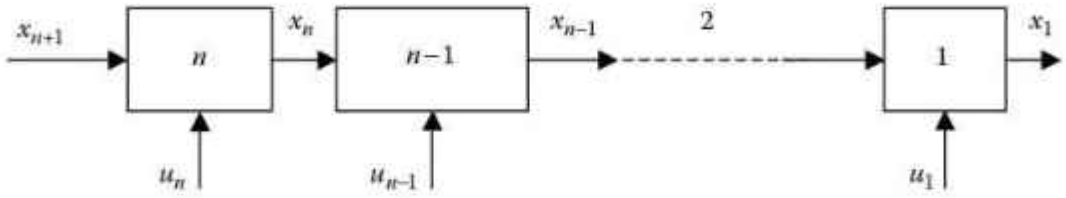


FIGURE 8.2
Multistage initial value problem.

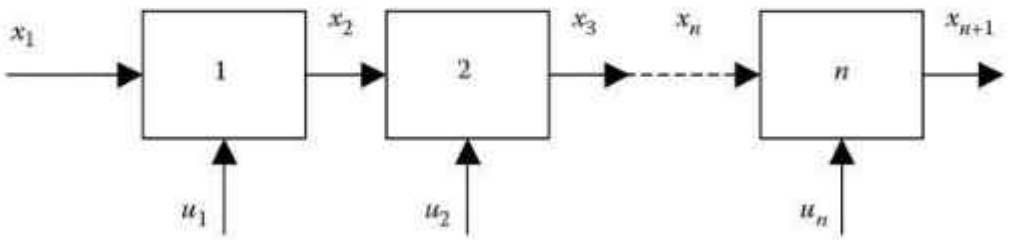


FIGURE 8.3
Multistage final value problem.

8.2.2 Types of Multistage Decision Problems

The serial multistage decision problems can be classified into the following categories.

1. Initial value problem: If the value of the initial state variable x_{n+1} is prescribed, the problem is called an initial value problem (Figure 8.2).
2. Final value problem: If the value of the final state variable x_1 is prescribed, the problem can be transformed into an initial value problem by reversing the directions of u_i, $i=1, 2, n+1$, which is shown in Figure 8.3.
3. Boundary value problem: If the values of both the input and output variables are specified, the problem is called a boundary value problem.

8.3 Characteristics of DP

DP has the following characteristics:

1. Divisible into stages, policy decision is requested at each stage.
2. Each stage has a number of associated states.
3. Efficient policy decision transformation of the current state is needed to associate the next stage.

4. Solution procedure is classified to find the optimum policy for the overall problem.
5. Prescription of optimum policy for the remaining stages, for each possible state.
6. Given the current stage, an optimal policy for the remaining stage is independent of the policy adopted in the previous stage (principle of optimality in DP).
7. By using the recursive relationship, the solution procedure moves backward stage by stage, each time finding the optimum policy for that stage until it finds the optimum policy starting at the initial stage.

8.4 Concept of Suboptimization and the Principle of Optimality

A DP problem can be stated as follows. Find the values of $u_1, u_2, \ldots, u_N$ which optimize $F = \sum_{i=1}^{n} f_i(u_i, x_{i+1})$ and satisfy the design equations $u_i = t_i(x_i, u_{i+1})$. DP makes use of the concept of suboptimization and Bellman's principle of optimality in solving the problem.

Bellman stated the following as part of his principle of optimality.

> An optimal policy (or a set of decisions) has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

In developing a recurrence relationship, suppose that the desired objective is to minimize the N-stage objective function f, which is given by the sum of the individual stage-returns:

$$\min \; F = f_n(x_{n+1}, u_n) + f_{n-1}(x_n, u_{n-1}) + \cdots + f_1(x_2, u_1) \tag{8.5}$$

and satisfying the design equations $x_i, = t_i(x_{i+1}, u_i)$. Figure 8.4 shows how to apply this to DP.

Now consider the first subproblem by starting at the final stage, $i = 1$. If the input to this stage x_2 is specified, then according to the principle of optimality, u_1 must be selected to optimize f_1. Irrespective of what happens to other stages, u_1 must be selected such that $f(u_1, x_2)$ is an optimum for the input x_2. If the optimum is denoted f_1^* we have

$$f_1^*(x_2) = \operatorname*{opt}_{u_1} [f_1(x_2, u_1)]. \tag{8.6}$$

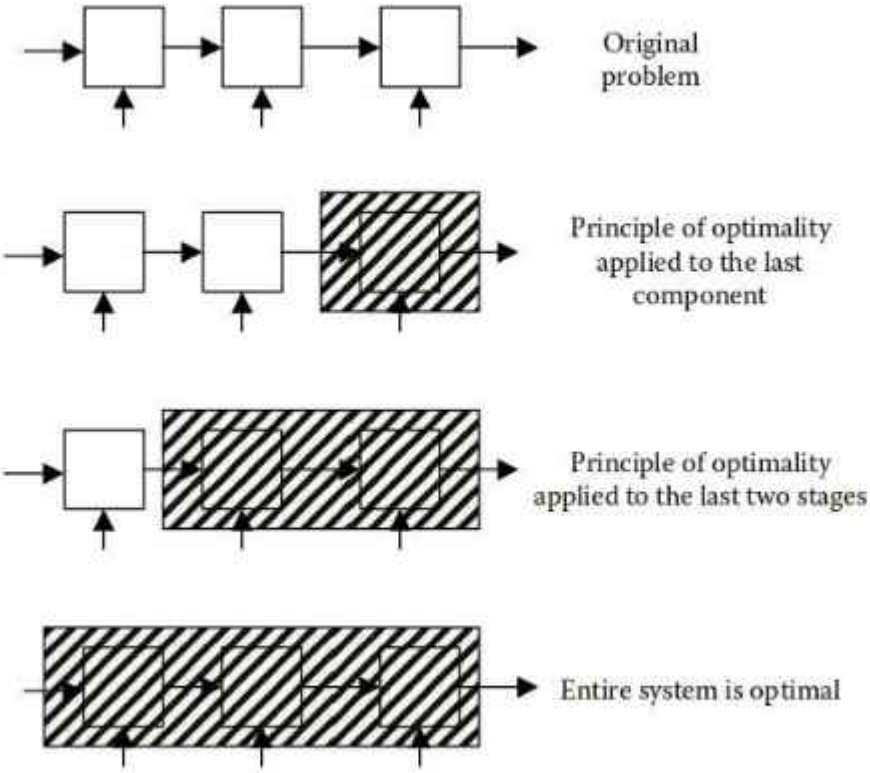


FIGURE 8.4
Illustration of the principle of optimality.

This is called a one-stage policy since once the input stage s_2 is specified, the optimal values of f_1, u_1, and x_2 are completely defined as shown in Equation 8.6.

Next, consider the second subproblem by grouping the last two stages together. If f_2^* denotes the optimum objective value of the second subproblem for a specified value of the input x_3, we have

$$f_2^*(x_3) = \underset{u_2,u_1}{\text{opt}} [f_1(x_2,u_1) + f_2(x_3,u_2)]. \tag{8.7}$$

The principle of optimality requires that u_1 be selected so as to optimize f_1 for a given x_2. Since x_2 can be obtained once u_2 and x_3 are specified, Equation 8.7 can be written as

$$f_2^*(x_3) = \underset{u_2}{\text{opt}} [f_1^* + f_2(x_3,u_2)]. \tag{8.8}$$

Thus f_2^* represents the optimal policy for the two-stage problem. Therefore, rewriting Equation 8.8 yields

$$f_2^*(x_3) = \underset{u_2}{\text{opt}} [f_1^*(t_2(u_2,x_3) + f_2(x_3,u_2))]. \tag{8.9}$$

In this form, it can be seen that for a specified input x_3, the optimum is determined solely by a suitable choice of the decision variable u_2. Thus the optimization problem stated in Equation 8.7, in which both u_2 and u_1 are to be simultaneously varied to produce the optimum f_2^*, is reduced to each of these subproblems involving only a single-decision variable, and optimization is, in general, much simpler.

This idea can be generalized and the ith subproblem defined by

$$f_i^*(x_{i+1}) = \underset{u_i,u_{i-1},\ldots,u_1}{\text{opt}} \lfloor f(u_i,x_{i+1}) + \cdots + f_1(x_2,u_1) \rfloor, \tag{8.10}$$

which can be written as

$$f_i^*(x_{i+1}) = \underset{u_i}{\text{opt}}[f_{i-1}(x_i) + f_i(x_{i+1},u_i)], \tag{8.11}$$

where

f_{i+1}^* denotes the optimal value of the objective function corresponding to the last $i-1$ stages

s_i is the input to the stage $i-1$

By using the principle of optimality, this problem has been decomposed into i separate problems, each involving only one decision variable.

8.5 Formulation of DP

DP is a transformation from a multiple vector decision process to a series of single-vector decision processes. It converts a problem that contains many vectors for decision processes but contains only one vector to decide at a time. The single vector is usually reduced to contain the minimal number of components for ease of optimization. DP not only reduces the complexity of problems, but also provides a form that can be solved effectively and iteratively on digital computers.

Consider an optimization problem that consists of n decision (control) vectors v_k for $k=1, 2, \ldots, n$ and a scalar objective function $P_n(v_n, v_n, v_{n-1}, \ldots, v_1)$ defined as state n. The vectors v_k have r_k components, which may change with the running index k. The decision vectors v_k are constrained in a set S, which is usually described by equations and decision vectors in S for a given state vector y_n.

Now, let $P_k(y_k, v_k, v_{k-1}, \ldots, v_1)$ be the objective function at stage k where $k=1, 2, \ldots, n$, and let

$$p_k(y_k) = \text{optimize}[P_k(y_k, v_k, v_{k-1}, \ldots, v_1)] = P_k(y_k, u_k, u_{k-1}, \ldots, v_1). \tag{8.12}$$

Equation 8.12 is the optimized objective function at stage k. The optimization is carried out with respect to v_1, v_2, v_k in S for a given y_k. The optimal decision or control vectors u, where $i=1, 2, \ldots, k$, are known vectors and hence the optimized objective function is a function of y_k only as indicated by the left side of Equation 8.12. As such, the problem is equivalent to determining Equation 8.12 for $k=n$. The state vectors y_k have n_k components, which may change with k. By properly choosing the integer r_k, we can always formulate the objective function as follows:

$$P_k(y_k, v_k, v_{k-1}, \ldots, v_1) = f_k(y_k, v_k) \circ P_{k-1}(y_{k-1}, v_k, v_{k-1}, \ldots, v_1), \tag{8.13}$$

for all $k=2, 3, 4, \ldots, n$ with $P_k(y_1, v_1) = f_1(y_1, v_1)$ for $k=1$.

The symbol "o" in Equation 8.13 is called an operator, which may be addition, multiplication, or comparison, and it may alter from stage to stage. The scalar function $f_k(y_k, v_k)$ is to be chosen in a manner such that it constitutes a part of the objective function. The relation specifies the state vector y_k,

$$y_{k-1} = T_k(y_k, v_k). \tag{8.14}$$

Both f_k and T_k denote a transformation or mapping that may be described in different forms. There are some known types of constraints that can be replaced by Equation 8.14. For example:

$$g_1(v_1) + g_2(v_2) + \cdots + g_n(v_n) \geq C \quad (\text{or } \leq C) \tag{8.15}$$

$$h_1(v_1)h_2(v_2), \ldots, h_n(v_n) \geq C \quad (\text{or } \leq C), \tag{8.16}$$

where g_s and h_s are scalar functions of the decision vectors. From Equation 8.15, we choose the state Equation 8.14 to be

$$y_{k-1} = y_k - g_k(v_k) \quad \text{for all } k = 1, 2, \ldots, n.$$

Summation of this equation gives

$$g_1(v_1) + g_2(v_2) + \cdots + g_n(v_n) = y_k - g_k(v_k).$$

Hence, $y_n \geq C$ can be concluded if $y_0 = 0$ is assumed.

Similarly, from Equation 8.16, we select the state equation to be

$$y_{k-1} = \frac{y_k}{h_k(v_k)},$$

where

$h_k(v_k) \neq 0$

$k=1, 2, \ldots, n$

Multiplication of these equations gives $h_1v_1,\ h_2v_2, \ldots,\ h_nv_n = y_n/y_0 \geq C$ and hence $y_n \geq C$ follows if $y_0 = 0$ is chosen. Thus, we may choose DP to get $P_n(y_n)$ and then make further optimization for $y_n \geq C\ (>C)$.

We call an optimization problem decomposable if it can be solved by recursive optimization through N stages, at each stage optimization being done over one decision variable. We first define monotonicity of a function, which is used subsequently.

DEFINITION 8.5.1

The function $f(x,y)$ is said to be a *monotonic nondecreasing function* of y for all feasible values of x if $y_1 < y_2 \Rightarrow f(x,y_1) \geq f(x,y_2)$ for every feasible value of x.

Conversely, the function is said to be *monotonic nonincreasing* if $y_1 < y_2 \Rightarrow f(x,y_1) \leq f(x,y_2)$ for every feasible x.

THEOREM 8.5.1

In a serial two-stage minimization or maximization problem if

(a) *Objective function ϕ_2 is a separable function of stage returns $f_1(X_1,U_1)$ and $f_2(X_2,U_{21})$, and*

(b) *ϕ_2 is a monotonic nondecreasing function of f_1 f_0 for every feasible value of f_2, then the problem is said to be decomposable.*

Proof Putting $N=2$ in the problem, the objective function $\phi_2(f_2,f_1)$ is separable if $\phi_2 = f_2 \circ \phi_1$, $\phi_1 = f_1$. Assuming that this condition holds true, then suppose that ϕ_2 is a monotonic nondecreasing function of f_1 for feasible values of f_2. We prove the theorem for the minimum case. For the maximum case the proof is on identical lines.

Following the notation introduced before, we note that the following expressions are equivalent.

$$F_2(X_2) = \min_{U_1,U_2} \phi_2(X_2,U_2,U_1) \tag{8.17a}$$

$$= \min_{U_1,U_2} \phi_2[f_2(X_2,U_2) \circ f_1(X_1,U_1)] \tag{8.17b}$$

$$= \min_{U_1,U_2} \phi_2[f_2(X_2,U_2) \circ f_1(X_1,U_2,U_1)]. \tag{8.17c}$$

The last form is possible because of the transformation relation

$$X_1 = t_2(X_2,U_2). \tag{8.18}$$

Also

$$F_2(X_2) = \min_{U_1} f_1(X_1,U_1) = \min_{U_1} f_1(X_1,U_2,U_1). \tag{8.19}$$

Let

$$F_2'(X_2) = \min_{U_2} [f_2(X_2,U_2) \text{ o } f_1(X_1)] \tag{8.20}$$

$$= \min_{U_2} \left[f_2(X_2,U_2) \text{ o } \min_{U_2} f_1(X_2,U_2,U_1) \right]. \tag{8.21}$$

Comparing Equations 8.17 and 8.21, we get $F_2'(X_2) \geq F_2(X_2)$.

Let

$$\min_{U_2} f_1(X_2,U_2,U_1) = f_1(X_2,U_2,U_1^0). \tag{8.22}$$

Then

$$f_1(X_2,U_2,U_1) \geq f_1(X_2,U_2,U_1^0). \tag{8.23}$$

Since ϕ_2 is a monotonic nondecreasing function of f_1, the above inequality implies

$$\begin{aligned} &\phi_2(X_2,U_2,U_1) \geq \phi_2(X_2,U_2,U_1^0) \\ &\phi_2(X_2,U_2,U_1) \leq \min_{U_1} \phi_2(X_2,U_2,U_1^0). \end{aligned} \tag{8.24}$$

Now from Equations 8.21 and 8.23, we observe that

$$\begin{aligned} F_2'(X_2) &= \min_{U_2} \left[f_2(X_2,U_2) \text{ o } f_1(X_2,U_2,U_1^0) \right] \\ &= \min_{U_2} \phi_2(X_2,U_2,U_1^0) \\ &\leq \min_{U_2} \min_{U_1} \phi_2(X_2,U_2,U_1^0) = sF_2(X_2). \end{aligned}$$

And from Equations 8.15 and 8.17, we get $F_2(X_2) = F_2'(X_2)$ or

$$F_2(X_2) = \min_{U_2} [f_2 \text{ o } F_1(X_1)].$$

The following theorem is an extension of the above to an N-stage optimization problem, treating stages $N-1$ to 1 as a single stage. Theorem 8.5.2 is a direct consequence of Theorem 8.5.1 and needs no further proof.

THEOREM 8.5.2

If the real-valued return function $\phi_N(f_N, f_{N-1}, \ldots, f_1)$ *satisfies*

(a) *Condition of separability, that is*

$$\phi_N(f_N, f_{N-1}, \ldots, f_1) = f_N \circ f_{N-1},$$

where $\phi_N(f_N, f_{N-1}, \ldots, f_1)$ *is real-valued, and*

(b) ϕ_2 *is a monotonic nondecreasing function of* ϕ_{N-1}, *for every* f_N: *then* ϕ_N *is decomposable: that is*

$$\min_{U_N, \ldots U_1} \phi_N(f_N, \ldots, f_1) = \min_{U_x} \left(f_N \circ \min_{U_N 1, \ldots, U_1} \phi_{N-1} \right).$$

Theorems 8.5.1 and 8.5.2 indicate that monotonicity is a sufficient condition for decomposability. The converse has not been proved. In fact the monotonicity condition is not necessary.

Often, it is tedious in applications to judge an objective function. Operators that belong to one of the following cases are monotonic and hence the decomposition theorem is applicable.

Case (a): All the operators are additions

Inequality (Equation 8.23) because

$$P_{k-1}(y_{k-1}) \geq P_{k-1}(y_{k-1}, v_{k-1}, v_{k-2}, \ldots, v_1)$$

for maximization, and the inequality reverses for minimization by definition. Note that abstractions can be converted to additions by absorbing the negative signs.

Case (b): All the operators are multiplications and $f_k(y_k, v_k) \geq 0$ for all $k = 1, 2, \ldots, n$.

The reason is the same as in case (a).

Case (c): *Combination of (a) and (b).*

The reason is the same as case (a).

Case (d): All the operators are comparison.

In such a case, the maximization is defined as follows.

$$P(y_k) = \max[f_k(y_k v_k), f_{k-1}(y_{k-1}, v_{k-1}), \ldots, f_1(y_1, v_1)],$$

and minimization is defined by replacing maximum by minimum. For maximization, we have $\max [f_k(y_k, v_k), p_{k-1}(y_{k-1})] = f_k$. When the first is greater than or equal to the second $= P_{k-1}(y_{k-1})$. Otherwise

$$\max[f_k(y_k, v_k), p_{k-1}(y_{k-1}, v_{k-1}, \ldots, v_1)] = f_k,$$

when the first is greater than or equal to the second, $p_{k-1}(y_{k-1}, v_{k-1}, \ldots, v_1)$ otherwise.

It follows from $P_{k-1}(y_{k-1}) \geq p_{k-1}(y_{k-1}, v_{k-1}, \ldots, v_1)$ that Equation 8.23 holds true. Similarly, Equation 8.23 can be shown valid for minimization.

The task of DP is to carry out the iterative process generated by the decomposition theorem for a monotonic objective function. The advantage of using DP is that only one decision vector is involved in each iteration. As a rule of thumb, it is desirable to reduce the integer r_k as small as possible without violating. This requires more stages but fewer variables in each iteration of the optimization.

8.6 Backward and Forward Recursion

Throughout we have used a recursion procedure in which x_j is regarded as the input and x_{j-1}, as the output for the nth stage, the stage returns are expressed as functions of the stage inputs, and recursive analysis proceeds from stage 1 to stage n. This procedure is called backward recursion because the stage transformation function is of the type $x_{j-1} = t_j(x_j, u_j)$. Backward recursion is convenient when the problem involves optimization with respect to a given input x_n, because then the output x_0 is very naturally left out of consideration.

If, however, the problem is to optimize the system with respect to a given output x_0, it would be convenient to reverse the direction, regard x_j as a function of x_{j-1} and u_j, and put $x_{j-1} = t_j(x_j, u_j)$, $1 \leq j \leq n$ and also express stage returns as functions of stage outputs and then proceed from stage n to stage 1. This procedure is called forward recursion.

In problems where both the input x_N and the output x_0 are given parameters, it is immaterial whether one proceeds in one direction or the other. Both parameters are retained during analysis, and the optimal solution is a function of both. In fact, for most multistage problems there is no essential difference between these two procedures. In mathematical problems inputs and outputs are fictitious concepts and are interchangeable. One can visualize and solve the problem in any direction. It only involves a slight modification in notation.

However, DP is also applicable to nonserial multistage systems, which are important in automatic control systems and in certain process technologies. In such systems stages are not all connected in series but with branches and loops. It is in the application of such systems that the difference between forward and backward recursion procedures becomes not only significant but also crucial. For this reason we proceed to write the recursion formulae for the forward procedure explicitly.

Assume that the return function $\phi_1(x_n, x_0, u_n, \ldots, u_j)$ is a function of the stage returns $f_j(x_j, x_{j-i} u_j)$ in the form

$$\phi_1 = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_j,$$

and further assume that the stage transformation function is given by

$$x_j = i_j(x_{j-1}x_j).$$

Then, by defining

$$F_j(x_{j-1}) = \min_{u_N,\ldots,u_f} (f_n \circ f_{n-1},\ldots, f_j)$$

we postulate the forward recursion formulae as

$$F_j(x_{j-1}) = \min_{u_j} \left[f_j(x_{j-1},u_j) \circ f_{j+1}(x_j) \right], \quad 1 \le j \le n-1$$

$$F_N(x_{n-1}) = f_n(x_{n-1},u_n).$$

With this notation the optimum value of ϕ_1 which we seek is denoted by $F_1(x_0)$ which is obtained recursively through stages $j = n-1,\ldots, 2, 1$.

8.6.1 Minimum Path Problem

Consider the following DP problem where we must find the shortest path from vertex A to vertex B along arcs joining the various vertices lying between A and B (Figure 8.5). The lengths of the paths are as shown.

The vertices are divided into five stages that we denote by subscript j. For $j=0$, there is only one vertex A; also for $j=4$ the only vertex is B. For $j=1,2,3$, there are three vertices in each stage. Each move consists of moving from stage j to stage $j+1$, that is, from any one vertex in stage j to any one vertex in stage $j+1$.

We say that each move changes the state of the system that we denote x_j, and x_0 is the state in which node A lies. Notably, x_0 has only one value,

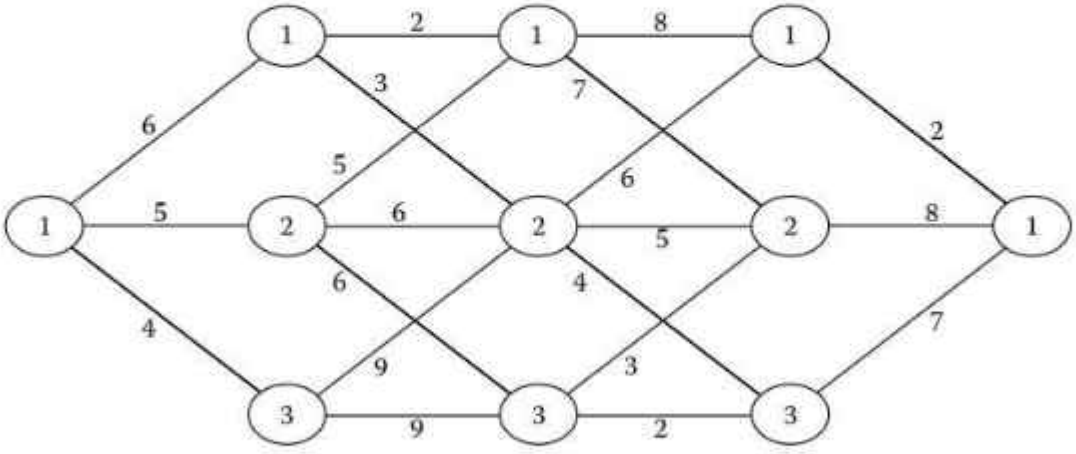


FIGURE 8.5
Minimum path problem.

say $x_0 = 1$. The state x_2 has three possible values, say 1, 2, 3, corresponding to three vertices in stage 2, and so on. We call the possible alternative paths from one stage to the next *decision* variables, and denote by u_j the decision variable that takes us from the state x_{j-1} to state x_j. The return or the gain from the decision u_j we denote *by* $f_j(u_j)$, return obviously being the function of the decision. In this problem we can identify u_j with the length of the corresponding arc, and so simplify matters by taking $f_j(u_j) = u_j$.

We denote by $F_j(x_j)$ the minimum path from the state, from x_0 any vertex in state x_j. Thus, $F_2(1)$ denotes the minimum path from A to vertex 1 of stage 2. The problem is to determine the minimum path $F_4(x_4)$, and the values of the decision variables u_1, u_2, u_3, and u_4 which yield that path. Let us look at the problem in the following way. The value of u_4 can either be 2, 8, or 7. If $u_4 = 2 \Rightarrow x_3 = 1$; similarly $u_4 = 8 \Rightarrow x_3 = 2$, and if $u_4 = 7 \Rightarrow x_3 = 3$, the minimum path from A to B is either through $x_3 = 1, 2$, or 3. With x_3 as 1, 2, or 3, the respective values of u_4 are 2, 8, or 7. Thus

$$F_4(x_4) = \min \begin{cases} 2 + F_3(1) \\ 8 + F_3(2) \\ 7 + F_3(3) \end{cases} = \min_{u4} (u_4 + F_3(x_3)).$$

Similarly we can argue that

$$F_3(1) = \min \begin{cases} 8 + F_2(1) \\ 6 + F_2(2) \end{cases},$$

or, in general

$$F_3(2) = \min \begin{cases} 7 + F_2(1) \\ 5 + F_2(2) \\ 3 + F_3(3) \end{cases}.$$

Hence

$$F_3(3) = \min \begin{cases} 4 + F_2(2) \\ 2 + F_2(3) \end{cases}.$$

Finally

$$F_3(x_3) = \min_{u_3} (u_3 + F_2(x_2)).$$

We have thus a general recursion formula

$$F_2(x_2) = \min_{u_2} (u_2 + F_1(x_1))$$

that enables us to determine $F_4(x_4)$ recursively.

As we later show, the number of enumerations gone through in this way substantially reduces the total number of enumerations that will have to be gone through if all possible paths are examined.

We now tabulate the information given in the problem and the enumerative steps necessary to find $F_4(x_4)$ with the help of the above recursion formula. We also simultaneously introduce the standard terminology of DP.

Tables 8.1 through 8.4 imply that a function of the type $x_{j-1} = t_j(x_j, u_j)$ exists. It follows from the data of the problem and we call it the stage

TABLE 8.1

Stage 1 for Minimum Path Problem

u_1	6	5	4
x_1	1	2	3

TABLE 8.2

Stage 2 for Minimum Path Problem

u_2 / x_2	x_1 = 2	3	5	6	9
1	1	—	2	—	—
2	—	1	—	2	3
3	—	—	—	2	3

TABLE 8.3

Stage 3 for Minimum Path Problem

u_3 / x_3	x_2 = 2	3	4	5	6	7	8
1	—	—	—	—	2	—	1
2	—	3	—	2	—	1	—
3	3	—	2	—	—	—	—

TABLE 8.4

Stage 4 for Minimum Path Problem

u_4 / x_4	x_3 = 2	7	8
1	1	3	2

transformation function. x_{j-1} may not be defined for all combinations of x_j, u_j. A dash in the tables indicates that the transformation for that pair of values x_j, u_j is not defined and therefore that transformation is not feasible. In Tables 8.5 through 8.8 the recursive operations using the recursive formulae are indicated.

The minimum path from A to B is thus found to be 17. Tracing the minimum path and decisions backwards, the successive decisions are 6, 3, 6, and 2 and the states are $x_0=1$, $x_1=1$, $x_2=2$, $x_3=3$, and $x_4=1$.

TABLE 8.5

Step 1 for the Recursive Operations

J	x_1	u_1	$F_1(x_1)$
	1	6	6
1	2	5	5
	3	4	4

TABLE 8.6

Step 2 for the Recursive Operations

u_2 / x_2	$F_1(x_1)$					$F_1(x_1)+u_2$					$F_2(x_2)$
	2	3	5	6	9	2	3	5	6	9	
1	6	—	5	—	—	8	—	10	—	—	8
2	—	6	—	5	4	—	9	—	11	13	9
3	—	—	—	5	4	—	—	—	11	13	11

TABLE 8.7

Step 3 for the Recursive Operations

u_3 / x_3	$F_2(x_2)$							$F_2(x_2)+u_3$							$F_3(x_3)$
	2	3	4	5	6	7	8	2	3	4	5	6	7	8	
1	—	—	—	—	9	—	8	—	—	—	—	15	—	16	15
2	—	11	—	9	—	8	—	—	14	—	14	—	15	—	14
3	11	—	9	—	—	—	—	13	—	13	—	—	—	—	13

TABLE 8.8

Step 4 for the Recursive Operations

u_4 / x_4	$F_3(x_3)$			$F_3(x_3)+u_4$			$F_4(x_4)$
	2	8	7	2	8	7	
1	15	14	13	17	22	20	17

8.6.2 Single Additive Constraint and Additively Separable Return Problem

Consider the following problem as another illustration of the method of DP.

Find the value of u_j that minimizes

$$z = \sum_{j=1}^{n} f_j(u_j),$$

where $1 \leq j \leq n$, subject to the constraints

$$\sum_{j=1}^{n} a_j u_j \geq b,\ a_j,\ b \in R,\ a_j \geq 0,\ b$$

$$u_j \geq 0.$$

The objective or return function z is a separable additive function of the n variable u_j. We look upon the problem as an n-stage problem, the suffix j indicating the stage. We have to decide about the values of u_j, and so the u_j is called decision variables. With each decision u_j is associated a return function $f_j(u_j)$ which is the return at the jth stage.

Now we introduce the variables $x_0, x_1, x_2, \ldots, x_n$ defined as follows:

$$\left.\begin{aligned} x_n &= a_1u_1 + a_2u_2 + \cdots + a_nu_n \geq b \\ x_{n-1} &= a_1u_1 + a_2u_2 + \cdots + a_{n-1}u_{n-1} = x_n - a_nu_n \\ x_{n-2} &= a_1u_1 + a_2u_2 + \cdots + a_{N-2}u_{n-2} = x_{n-1} - a_{n-1}u_{n-1} \\ &\vdots \qquad\qquad\qquad \vdots \\ x_1 &= a_1u_1 = x_2 - a_2u_2 \end{aligned}\right\}. \tag{8.25}$$

These variables are called the state variables. We further notice that $x_{j-1} = t_j(x_j, u_j)$ for $1 \leq j \leq n$. That is, each state variable is a function of the next state and decision variable. This is the stage transformation function. Since x_n is a function of all the decision variables we may denote by $F_n(x_n)$ the minimum value of u for any feasible function of x_n,

$$F_n(x_n) = \min_{u_1, u_2, \ldots, u_n} [f_1(u_1) + f_2(u_2) + \cdots + f_n(u_n)], \tag{8.26}$$

the minimization being over nonnegative values of u_j subject to $x_n \geq b$.

We select a particular value of u_n and holding u_n fixed, we minimize u over the remaining variables. This minimum is given by

$$f_n(u_n) + \min_{u_1, u_2, \ldots, u_{n-1}} [f_1(u_1) + f_2(u_2) + \cdots + f_{n-1}(u_{n-1})] = f_n(u_n) + F_{n-1}(x_{n-1}). \tag{8.27}$$

And the values of $u_1, u_2, \ldots, u_{n-1}$ that would make $\sum_{j=1}^{n-1} f_j(u_j)$ minimum for a fixed u_n thus depending upon x_{n-1} which in turn is a function of x_n and u_n. Also, the minimum u over all u_n for feasible x_n would be

$$F_n(x_n) = \min_{u_n} [f_n(u_n) + F_{n-1}(x_{n-1})]. \tag{8.28}$$

If somehow $F_{n-1}(x_{n-1})$ were known for all u_n, the above minimization would involve a single variable u_n. Repeating the argument, we get the recursion formula

$$\begin{aligned} F_j(x_j) &= \min_{u_j} \left[f_j(u_j) + f_{j-1}(x_{j-1}) \right], \quad 2 \le j \le n \\ F_1(x_1) &= f_1(u_1), \end{aligned} \tag{8.29}$$

which along with the relation $x_{j-1} = t_j(x_j, u_j)$ defines a typical DP problem.

If we could make a start with $F_1(x_1)$ and recursively go on to optimize to get $F_2(x_2)$, $F_3(x_3), \ldots, F_n(x_n)$, each optimization being done over a single variable, we would get $F_n(x_n)$ for each possible x_n. Minimizing it over x_n we get the solution. The following numerical examples illustrate how this can be done.

8.6.3 Single Multiplicative Constraint, Additively Separable Return Problem

Consider the problem

$$\min \sum_{j=1}^{n} f_j(u_j)$$

subject to the constraints

$$\prod_{j=1}^{n} u_j \ge k > 0, \quad u_j \ge 0.$$

We introduce the state variables x_j defined as follows.

$$\left.\begin{aligned} x_n &= u_n u_{n-1} = \cdots = u_2 u_1 \ge k, \\ x_{n-1} &= x_n / u_n = u_{n-1} \ldots u_2 u_1, \\ &\ldots \\ x_2 &= x_3 / u_2 = u_2 u_1, \\ x_1 &= x_2 / u_2 = u_1 \end{aligned}\right\}. \tag{8.30}$$

These are the stage transformations of the type

$$x_{j-1} = t_j(x_j, u_j).$$

Denoting by $F_n(x_n)$ the minimum value of the objective function for any feasible x_n, we can get the recursion formula

$$\begin{aligned} F_j(x_j) &= \min_{u_j} \left[f_j(u_j) + f_{j-1}(x_{j-1}) \right], \quad 2 \le j \le n \\ F_1(x_1) &= f_1(u_1) \end{aligned} \tag{8.31}$$

which will lead to the solution.

Example 8.6.1 (see Tables 8.9 and 8.10)

Maximize $u_1^2 + u_2^2 + u_3^2$

Subject to

$u_1 u_2 u_3 \le 6$

and

u_1, u_3, u_3 are positive integers.

SOLUTION

The state variables are

$$x_3 = u_1 u_2 u_3 \le 6, \quad x_2 = x_3/u_3 = u_1 u_2, \quad x_1 = x_2/u_2 = u_1.$$

TABLE 8.9

Stage 1 for Example 8.6.1

u_j	1	2	3	4	5	6
$f_j(u_j)$	1	4	9	16	25	36

TABLE 8.10

Stage 2 for Example 8.6.1

	u_j					
x_j	**1**	**2**	**3**	**4**	**5**	**6**
1	1	—	—	—	—	—
2	2	1	—	—	—	—
3	3	—	1	—	—	—
4	4	2	—	1	—	—
5	5	—	—	—	1	—
6	6	3	2	—	—	1

The solution is worked out as follows:

Stage return

$$f_j(u_j) = u_j^2; \quad j = 1, 2, 3.$$

Stage transformation

$$x_{j-1}(u_j), \quad j = 2, 3.$$

Recursion operation (see Tables 8.11 through 8.13)

The answer is 38 with $u_3 = 1$, $u_2 = 1$, $u_1 = 6$.

TABLE 8.11

Step 1 for the Recursive Operation

x_1	1	2	3	4	5	6
$F_1(x_1)$	1	4	9	16	25	36

TABLE 8.12

Step 2 for the Recursive Operation

u_2 / x_2	$f_2(u_2)$						$F_1(x_1)$						
	1	2	3	4	5	6	1	2	3	4	5	6	$F_2(x_2)$
1	1	—	—	—	—	—	1	—	—	—	—	—	2
2	1	4	—	—	—	—	4	1	—	—	—	—	5
3	1	—	9	—	—	—	9	—	1	—	—	—	10
4	1	4	—	16	—	—	16	4	—	1	—	—	17
5	1	—	—	—	25	—	25	—	—	—	1	—	26
6	1	4	9	—	—	36	36	9	4	—	—	1	37

TABLE 8.13

Step 3 for the Recursive Operations

u_3 / x_3	$f_3(u_3)$						$F_2(x_2)$						
	1	2	3	4	5	6	1	2	3	4	5	6	$F_3(x_3)$
1	1	—	—	—	—	—	2	—	—	—	—	—	3
2	1	4	—	—	—	—	5	2	—	—	—	—	6
3	1	—	9	—	—	—	10	—	2	—	—	—	11
4	1	4	—	16	—	—	17	5	—	2	—	—	18
5	1	—	—	—	25	—	26	—	—	—	2	—	27
6	1	4	9	—	—	36	37	10	5	—	—	2	38

8.6.4 Single Additive Constraint, Multiplicatively Separable Turn Problem

Consider the problem:

$$\text{Maximize } \prod_{j=1}^{n} f_j(u_j) \tag{8.32}$$

Subject to

$$\sum_{j=1}^{n} a_j u_j = k \tag{8.33}$$

and

$$u_j \geq 0, \quad a_j \geq 0.$$

Here, our state variables are

$$x_n = \sum_{j=1}^{n} a_j u_j = k,$$

$$x_{j-1} = x_j - a_j u_j, \quad 2 \leq j \leq n.$$

Putting

$$F_j(x_j) = \max_{u_1, u_2, \ldots, u_j} \prod_{j=1}^{n} f_j(u_j), \quad 2 \leq j \leq n,$$

the general recursion formula is

$$\begin{aligned} F_j(x_j) &= \max_{u_j} \prod_{j=1}^{n} \left[f_j(u_j) F_{j-1}(x_{j-1}) \right], \quad j - n, n-1, \ldots, 2 \\ F_1(x_1) &= f_1(u_1). \end{aligned} \tag{8.34}$$

8.7 Computational Procedure in DP

We now discuss the use of the recurrence relationships. DP begins by sub-optimizing the fast component which means the determination of

$$f_1^*(x_2) = \operatorname*{opt}_{u_1} [F_1(u_1, x_2)].$$

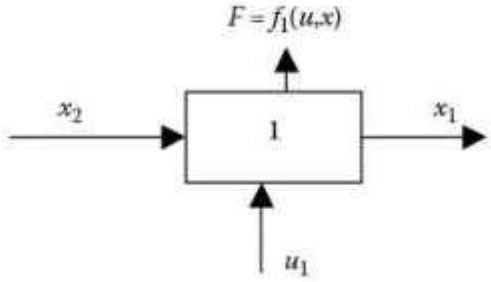


FIGURE 8.6
Suboptimization problem of component 1 for various settings of input state variable x_2.

The best value of the decision variable u_1, denoted u_1^*, is that which makes the objective function f_1, denoted f_1^*, hit its maximum value. Both depend on the condition of the input or feed that component 1 receives from upstream. Figure 8.6 shows a summary of the suboptimization problem of stage 1.

Next, we move up the serial system to include the fast two components. In this two-stage suboptimization, we have to determine

$$f_2^*(x_3) = \underset{u_2,u_1}{\text{opt}}\,[F_1(u_1,x_2) + F_2(u_2,x_3)]$$

since all the information about the first stage has already been calculated. Then the result can be substituted to get the following equation for the second stage

$$f_2^*(x_3) = \underset{u_2}{\text{opt}}\,\lfloor f_1^*(x_2) + F_2(u_2,x_3)\rfloor,$$

assuming the sequence has been carried on to include $(i-1)$ at the end of the components. Then the solution will be the solution of

$$f_i^*(x_{i+1}) = \underset{u_i,u_{i-1},\ldots,u_1}{\text{opt}}\,[F_i + F_{i-1} + \cdots + F_1].$$

However, all the information regarding the suboptimization of $(i-1)$ end components is known and has been entered in the table corresponding to f_{i-1}^*. Then we can obtain

$$f_1^*(x_{i+1}) = \underset{u_i}{\text{opt}}\,[F_i(u_i,x_{i+1}) + f_{i-1}(x_i)];$$

thus the dimension of the ith stage has been reduced to one.

8.8 Computational Economy in DP

DP for discrete problems is an enumerative procedure. A number of alternatives are examined at each stage and some selected for further examination. For large-scale problems, the number of alternatives is very large. Should we examine all the possible alternatives? The question now is if there is any saving in DP. And if so what is the order of the saving.

Consider a problem with n stages and let each decision variable u_j have p possible values and also each state variable u_j have p feasible values and an additive return function. In the direct exhaustive search, a feasible solution is specified by an input value x_n and the values of the decision variables u_j $(j=1, 2, \ldots, n)$ and each component has p values. Then the total number of feasible solutions is p^{n+1}. To get the objective from each feasible solution, we have to add two at a time involving $(n-1)$ additions; it means $(n-1)p^{n+1}$ additions. Finally to choose the optimum of these we have to make (p^{n+1}) comparisons. The total number of steps is

$$N(\text{DS}) = (n-1)p^{n+1} + p^{n+1} - 1 = np^{n+1} - 1. \tag{8.35}$$

But for DP, for stages starting at the second one up to n, for every combination, p^2 combinations per stage, we need only one addition. The total number of additions is $(n-1)p^2$ additions. Also at each stage starting from the first one, for each value of x_j, p numbers have to be compared in $(p-1)$ comparisons giving a total of $[np(p-1)]$. To get the optimum value, p numbers have to be compared. Then the total number of additions and comparisons in DP are

$$N(\text{DP}) = (n-1)p^2 + np(p-1) + p - 1 = (2n-1)p^2 - (n-1)p - 1, \tag{8.36}$$

which is much less than in the exhaustive search case.

8.9 Systems with More than One Constraint

DP can be applied to problems involving many constraints. In multiconstraint problems, there has to be one state variable per constraint per stage. Sometimes it is possible to take advantage of the problem structure to return the number of state variables.

Example 8.9.1

Maximize $u_1^2 + u_2^2 + u_3^2$

Subject to

$u_1u_2u_3 \leq 6, \quad u_1 + u_2 + u_3 \leq 6$

and u_1, u_2, u_3 are positive integers.

SOLUTION

The two sets of state variables are

$$\begin{aligned} x_3 &= u_1u_2u_3 & y_3 &= u_1 + u_2 + u_3 \\ x_2 &= x_3/u_3 = u_1u_2 & y_2 &= y_3 - u_3 = u_1 + u_2 \\ x_1 &= x_2/u_2 = u_1 & y_1 &= y_2 - u_2 = u_1. \end{aligned}$$

The feasible values of u_j are 1, 2, 3, 4. For stage $j=1$, the stage transformation gives the following possible values of x_1, y_1.

$$\begin{array}{llllll} u_1 & 1 & 2 & 3 & 4, \\ x_1 & 1 & 2 & 3 & 4, \\ y_1 & 1 & 2 & 3 & 4. \end{array}$$

For $j=2, 3$, Table 8.14 gives the transformations

$$x_{j-1} = t_{j-1}(x_j,u_j), \quad y_{j-1} = t'_{j-1}(y_j,u_j).$$

Because of the constraints we do not need to consider $x_j, y_j > 6$ (see Tables 8.15 through 8.17). Hence max $F_3(x_3,y_3) = 18$ for $(x_3,y_3) = (4,6)$. Tracing back, the optimal decision variables are (1, 1, 4) or (1, 4, 1) or (4, 1, 1).

TABLE 8.14

Transformation for Example 8.9.1

u_j / x_iy_j	x_{j-1}, y_{j-1}			
	1	**2**	**3**	**4**
1,1	1,0	—,—	—,—	—,—
2,2	2,1	1,0	—,—	—,—
3,3	3,2	—,1	1,0	—,—
4,4	4,3	2,2	—,1	1,0
5,5	5,4	—,3	—,2	—,1
6,6	6,5	3,4	2,3	—,2

TABLE 8.15

Stage 1 of Optimization of Example 8.9.1

u_1	X_1	Y_1	$F(x_1,y_1)$
1	1	1	1
2	2	2	4
3	3	3	9
4	4	4	16

TABLE 8.16

Stage 2, $F_2(X_2,Y_2) = \min_{U_2} \left[U_2^2 + F_1(X_1,Y_1)\right]$

u_2	x_2	y_2	$F_1(x_1,y_1)$	$u_2^2 + F_1(x_1,y_1)$	x_2	y_2	$F_2(x_2,y_2)$
	1	1	1	2	1	2	2
	2	2	4	5	2	3	5
1	3	3	9	10	3	4	10
	4	4	16	17	4	5	17
	1	1	1	5	2	3	—
2	2	2	4	8	4	4	8
	3	3	9	13	6	5	13
3	1	1	1	10	3	4	—
	2	2	4	13	6	5	—
4	1	1	1	17	4	5	—

TABLE 8.17

Stage 3, $F_3(x_3,y_3) = \max_{U_3} \left[u_3^2 + F_2(x_2,y_2)\right]$

u_3	x_2	y_2	$F_2(x_2,y_2)$	$\left[u_3^2 + F_2(x_2,y_2)\right]$	x_3	y_3	$F_3(x_3,y_3)$
	1	2	2	3	1	3	3
	2	3	5	6	2	4	6
	3	4	10	11	3	5	11
1	4	4	8	9	4	5	9
	4	5	17	18	4	6	18
	6	5	13	14	6	6	14
	1	2	2	6	2	4	—
2	2	3	5	9	4	5	—
	3	4	10	14	6	6	—
3	1	2	2	11	3	5	—
	2	3	5	14	6	6	—
4	1	2	2	18	4	6	—

8.10 Conversion of a Final Value Problem into an Initial Value Problem

Before we discussed DP with reference to an initial value problem. But if the problem is a final value problem as shown in Figure 8.7, it can be converted into an equivalent initial value problem. If the state transformation was given by Equation 8.1

$$x_i = t_i(x_{i+1}, u_i) \quad i = 1, 2, \ldots, n.$$

Assuming the inverse relations exist, we can write

$$x_{i+1} = \bar{t}_i(x_i, u_i) \quad i = 1, 2, \ldots, n,$$

where the input state to stage i is expressed as a function of its output state and its decision variable. Also if the objective function was originally expressed as

$$F_i = f_i(x_{i+1}, u_i) \quad i = 1, 2, \ldots, n,$$

it can be used to express it in terms of the output stage and the decision variable as

$$F_i = f_i[\bar{t}_i(x_i, u_i), x_i] = \bar{f}_i(x_i u_i), \quad i = 1, 2, \ldots, n.$$

Then we can use the same original approach as before in this new problem.

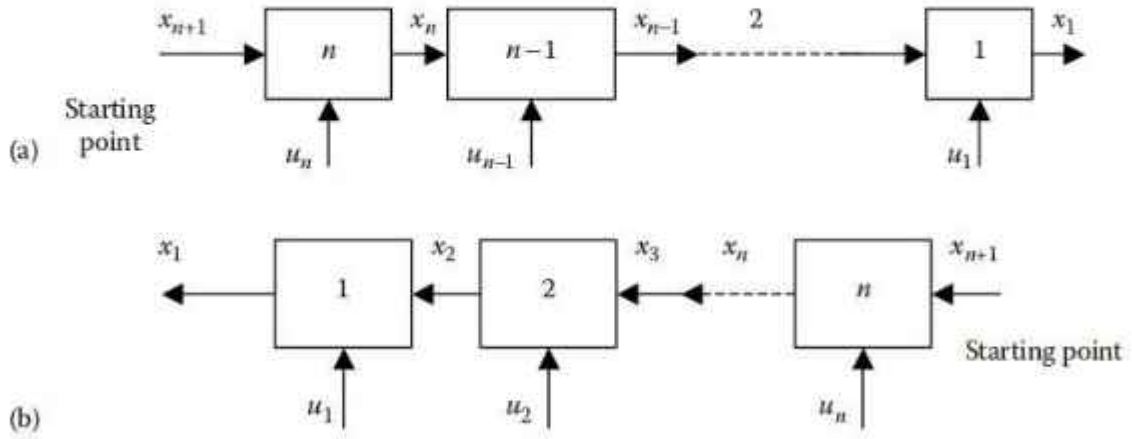


FIGURE 8.7
Conversion of (a) final value problem to (b) initial value problem.

8.11 Illustrative Examples

Example 8.11.1

Maximize $x_1^2 + x_2^2 - x_3^2$

Subject to

$x_1 + x_2 + x_3 \geq 14$

Put

$$\begin{aligned} u_3 &= x_1 + x_2 + x_3 \\ u_2 &= x_1 + x_2 = u_3 - x_3 \\ u_1 &= x_1 = u_2 - x_2 \\ F_3(u_3) &= \max_{x_3} \left[-x_3^2 + F_2(u_2)\right] \\ F_2(u_2) &= \max_{x_2} \left[x_2^2 + F_1(u_1)\right] \\ F_1(u_1) &= \left[x_1^2\right] = (u_2 - x_2)^2. \end{aligned}$$

Substituting in $F_2(u_2) = \max_{x_2}\left[x_2^2 + (u_2 - x_2)^2\right]$, by calculus, a function is a maximum if its partial differential equals zero. $F_2(u_2) = \left[x_2^2 + (u_2 - x_2)^2\right]$ is maximum if

$$\frac{\partial F_2(u_2)}{\partial x_2} = [2x_2 - 2(u_2 - x_2)] = 0 \quad \text{or} \quad x_2 = 2u_2.$$

Hence $F_2(u_2) = 5u_2^2$. Now

$$\begin{aligned} F_3(u_3) &= \max_{x_3} \left[-x_3^2 + F_2(x_2)\right] \\ &= \max_{x_3} \left[-x_3^2 + 5u_2^2\right] \\ &= \min_{x_3} \left[-x_3^2 + 5(u_3 - x_3)^2\right]. \end{aligned}$$

Again, by calculus, a function is maximum, if its partial differentiation equals zero.

$$F_3(u_3) = \left[-x_3^2 + 5(u_3 - u_3)^2\right]$$

is maximum if

$$\frac{\partial F_3(u_3)}{\partial x_3} = [-2x_3 - 10(u_3 - x_3)] = 0 \quad \text{or} \quad x_3 = \frac{5}{4}u_3.$$

Hence

$$F_3(u_3) = \left(\frac{-5}{4}\right)u_3^2, \quad u_3 \geq 14.$$

Obviously $F_3(x_3)$ is the maximum for $u_3 = 14$ which means that the maximum value of $(x_1^2 + x_2^2 - x_3^2)$ is therefore (-245). Back substitution to calculate the x_1, x_2, and x_3 gives

$$x_3 = \left(\frac{5}{4}\right)u_s = \left(\frac{5}{4}\right)(14) = 17.5, \quad u_2 = u_3 - x_3 = 14 - 17.5 = -3.5$$

$$x_2 = 2u_2 = -7, \quad u_1 = u_2 - x_2 = 3.5 = x_1.$$

Then the final result is $f_{max} = (-245)$, $x_1 = (3.5)$, $x_2 = (-7)$, and $x_3 = (17.5)$.

Example 8.11.2

The network shown in Figure 8.8 illustrates a transmission model with node (1) representing the generation unit, and nodes (2)–(7) representing load centers. The values associated with branches are power losses.

1. Derive an optimal policy of supplying the load at node (8) from the generation node (1) with minimum losses.
2. Derive an optimal policy of supplying the load at node (8) from the generation node (1) with minimum losses at node (1), but also supplying a load at node (4).

Solution

1. This is a minimization problem,

$$F_i(x_j) = \min_{u_j} \left\lfloor u_j + F_{j-1}(x_{j-1}) \right\rfloor,$$

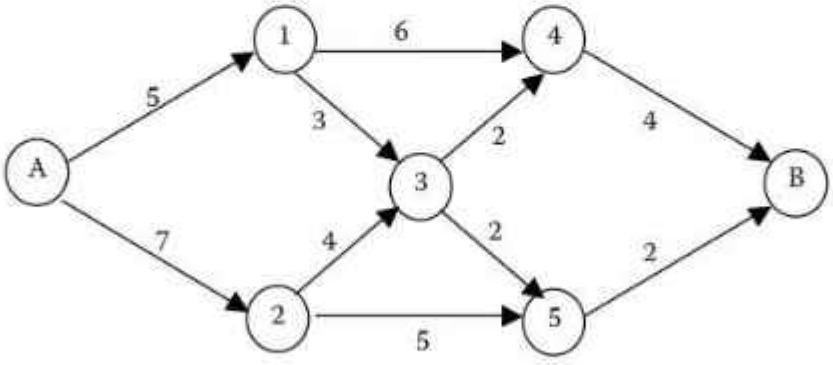


FIGURE 8.8
Transmission model for Example 8.11.2.

where, u_j is the connection from stage $(j-1)$ to stage j, based on the fact that $F_1(u_1) = u_1$. Then, $F_1(2) = 3$ and $F_1(3) = 5$.

$$F_2(x_2) = \min_{u_2} [u_2 + F_1(x_1)] = \min[F_2(4), F_2(5)]$$
$$F_2(4) = \min[10 + F_1(2), 11 + F_1(3)] = \min[10 + 3, 11 + 5] = 13$$
$$F_2(5) = \min[4 + f_1(3), 12 + F_1(2)] = \min[4 + 5, 12 + 3] = 9.$$

Then, $F_2(x_2) = \min[F_2(4), F_2(5)] = \min[13,9] = 9$. In a similar way,

$$F_3(x_3) = \min_{u_3} [u_3 + F_2(x_2)] = \min_{u_3} [F_3(6), F_3(7)].$$
$$F_3(6) = \min[15 + F_2(4), 10 + F_2(5)] = \min[15 + 13, 10 + 9] = 19$$
$$F_3(7) = \min[9 + F_2(4), 16 + F_2(5)] = \min[9 + 13, 16 + 9] = 22.$$

Then

$$F_3(x_3) = \min[F_3(6), F_3(7)] = \min[19, 22] = 19$$
$$F_4(x_4) = \min_{u_4} [u_4 + F_3(x_3)] = \min[9 + F_3(6), 8 + F_3(7)]$$
$$= \min[(9 + 19), (8 + 23)] = 29.$$

Then the minimum route is {node (1) – node (3) – node (4) – node (6) – node (8)}, and the value of the losses is (28).

2. We must pass by the node (4). Then the system is divided into two separate problems omitting node (5) as shown in Figure 8.9.

For the first part of the system:

$$F_1(u_1) = u_1.$$

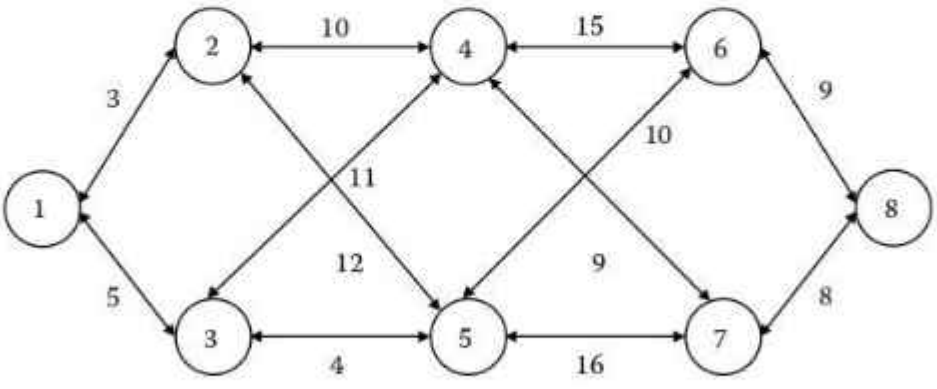


FIGURE 8.9
Reduced transmission model.

Then

$$F_1(2) = 3 \quad \text{and} \quad F_1(3) = 5$$

$$F_2(4) = \min_{u_2} [u_2 + F_1(x_1)] = \min[10 + F_1(2),\ 11 + F_1(3)]$$
$$= \min[13,16] = 13.$$

For the second part of the system:

$$F_3(x_3) = \min_{u_3} [u_3 + F_2(x_2)] = \min_{u_3} [F_3(6),\ F_3(7)]$$
$$= \min[(13 + 15),\ (13 + 9)] = 22$$
$$F_4(x_4) = \min_{u_4} [u_4 + F_3(x_3)] = \min[9 + F_3(6),\ 8 + F_3(7)]$$
$$= \min[(9 + 28),\ (8 + 22)] = 30.$$

Then the minimum route is {node (1) – node (2) – node (4) – node (7) – node (8)}, and the value of the losses is (30).

Example 8.11.3

A computer company has accepted a contract to supply 100 computers at the end of the first month and 150 at the end of the second month. The cost of manufacturing a computer is given by $\$(70x + 0.2x^2)$ where x is the number of manufactured units in that month. If the company manufactures more computers than needed in the first month, there is an inventory carrying charge of \$80 for each unit carried over to the next month. Find the number of computers manufactured in each month to maintain the cost at a minimum level, assuming that the company has enough facilities to manufacture up to 250 computers per month.

Solution

The total cost is composed of the production cost and the inventory carrying costs. The constrained optimization problem can be stated as follows:

Minimize: $f(x_1,x_2) = (70x_1 + 0.2x_1^2) + (70x_2 + 0.2x_2^2) + 80(x_1 - 80)$

Subject to

$x_1 \geq 100, \quad x_1 + x_2 = 250$

and x_1, x_2 are positive integers,

where x_1, x_2 denote the number of computers manufactured in the first and second months, respectively. This problem can be considered as a two-stage decision problem as shown in Figure 8.10.

Assuming the optimal solution of the first month equals $x_1^* = 100 + u_2$, the objective of the first month will be

$$f_1^* = 70(100 + u_1) + 0.2(100 + u_1)^2 = 9000 + 110u_1 + 0.2u_1^2.$$

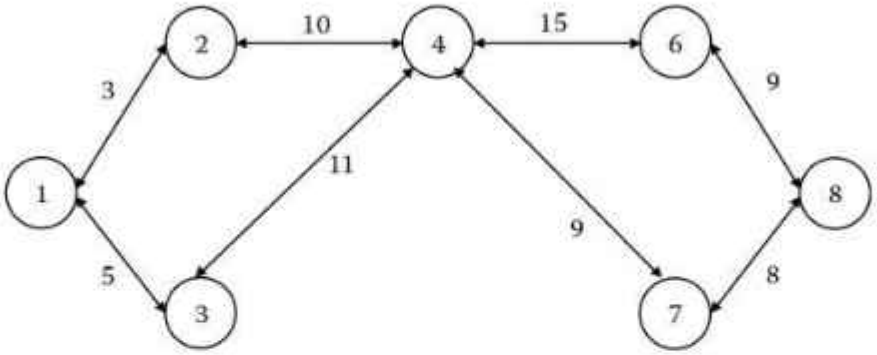


FIGURE 8.10
Two-stage decision problem.

The cost incurred in the second month is given by

$$f_2(x_2,u_1) = 70x_2 + 0.23x_2^2 + 10u_1.$$

The total cost will be

$$\begin{aligned} F(x_1,x_2,u_1) &= 9000 + 110u_1 + 0.2u_1^2 + 70x_2 + 0.2x_2^2 + 10u_1 \\ &= 9000 + 120u_1 + 0.2u_1^2 + 70x_2 + 0.2x_2^2. \end{aligned}$$

But the amount of inventory at the beginning of the second month plus the production in the second month must be equal to the supply in the second month. We have

$$x_2 + u_1 = 150 \Rightarrow u_1 = 150 - x_2.$$

Substituting for u_1 in the total cost function, we get

$$\begin{aligned} F &= 9000 + 120(150 - x_2) + 0.2(150 - x_2)^2 + 70x_2 + 0.2x_2^2 \\ &= 31{,}500 - 110x_2 + 0.4x_2^2, \end{aligned}$$

since this last equation is a function only in x_2. Then we can get the optimum value by setting

$$\frac{\partial F}{\partial x_2} = 0.0 \Rightarrow 110 = 0.8x_2 \Rightarrow x_2^* = 138.$$

Checking the second derivative,

$$\frac{\partial^2 F}{\partial x_2^2} = 0.8 > 0.$$

Then the second month's production of 80 corresponds to the minimum cost. The first month's production and the inventory will be $x_1^* = 112$, $u_1 = 12$ and the minimum total cost equals ($23937.6).

Example 8.11.4

$$\text{Max } P = (4d_1 - d_1^2) + (4d_2 - 2d_2^2) + (4d_3 - 3d_3^2) + (4d_4 - 4d_4^2)$$

$$\text{s.t. } d_1 + d_2 + d_3 + d_4 = c$$

$$d_k \geq 0, \quad k = 1, \ldots, 4$$

$$\text{Set } u_4 = d_1 + d_2 + d_3 + d_4$$

$$u_3 = d_1 + d_2 + d_3 = u_4 - d_4$$

$$u_2 = d_1 + d_2 = u_3 - d_3$$

$$u_1 = d_1 = u_2 - d_2$$

$$F_4(u_4) = \max_{d_4} \left[4d_4 - 4d_4^2 + F_3(u_3)\right]$$

$$F_3(u_3) = \max_{d_3} \left[4d_3 - 3d_3^2 + F_2(u_2)\right]$$

$$F_2(u_2) = \max_{d_2} \left[4d_2 - 2d_2^2 + F_1(u_1)\right]$$

$$F_1(u_1) = \left[4d_1 - d_1^2\right] = 4u_2 - 4d_2 - u_2^2 + 2u_2 d_2 - d_2^2$$

Substituting $F_2(u_2) = \max_{d_2} \left[-3d_2^2 + 4u_2 - u_2^2 + 2u_2 d_2\right]$

For $(\partial^2 F_2(u_2)/\partial u d_2)|_{\max} = 0 = -6d_2 + 2u_2$, We obtain $d_2 = (1/2)u_2$

Hence

$$F_2(u_2) = -3\left(\frac{1}{3}u_2\right)^2 + 4u_2 - u_2^2 + 2u_2\left(\frac{1}{3}u_2\right) = 4u_2 - \frac{2}{3}u_2^2$$

$$F_3(u_3) = \max_{d_3}\left[4d_3 - 3d_3^2 + 4u_2 - \frac{2}{3}u_2^2\right] = \max_{d_3}\left[-3\frac{2}{3}d_3^2 + 4u_3 - \frac{2}{3}u_3^2 + \frac{4}{3}u_3 d_3\right]$$

For $\max(\partial^2 F_3/\partial d_3) = (22/3)d_3 + (4/3)u_3 = 0$, we obtain $d_3 = (2/11)u_3$

Hence

$$F_3(u_3) = -\frac{11}{3}\left(\frac{2}{11}u_3\right)^2 + 4u_3 - \frac{2}{3}u_3^2 + \frac{4}{3}u_3\left(\frac{2}{11}u_3\right) = 4u_3 - 0.303u_3^2$$

And

$$F_4(u_4) = \max_{d_4}\left[4d_4 - 4d_4^2 + 4u_3 - 0.303u_3^2\right]$$

$$= \max_{d_4}\left[-4.303d_4^2 + 4u_4 - 0.303u_4^2 + 0.606u_4 d_4\right]$$

For $\left(\partial^2 F_4/\partial d_4\right) = -8.606d_4 + 0.606u_4 = 0$, we obtain $d_4 = 0.07u_4$

Hence $F_4(u_4) = -4.303(0.07u_4)^2 + 4u_4 - 0.303(u_4^2 + 6.06u_4(0.07u_4))$

$$= 0.1002u_4^2 + 4u_4$$

Let $u_4 = c$, maximum value of $P = 0.1002c^2 + 4c$, back substituting to calculate d_1, d_2, d_3, and d_4 gives,

$$d_4 = 0.07c, \quad u_3 = u_4 - d_4 = 0.93c$$

$$d_3 = \frac{2}{11}u_3 = 0.169c, \quad u_2 = u_3 - d_3 = 0.761c = 0.761c$$

$$d_2 = \frac{1}{3}u_2 = 0.254c, \quad u_1 = u_2 - d_2 = 0.507c$$

$$d_1 = u_1 = 0.507c$$

SOLUTION

$$P_{\max} = 0.1002c^2 + 4c, \quad \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 0.507c \\ 0.254c \\ 0.169c \\ 0.07c \end{bmatrix} = c\begin{bmatrix} 0.507 \\ 0.254 \\ 0.169 \\ 0.07 \end{bmatrix}.$$

Example 8.11.5

Solve Example 8.11.1 completely

Max $x_2^1 + x_2^2 - x_3^2$

s.t. $x_1 + x_2 + x_3 \geq 14$

Put $u_3 = x_1 + x_2 + x_3$

$u_2 = x_1 + x_2 = u_3 - x_3$

$u_1 = x_1 = u_2 - x_2$

$F_3(u_3) = \max_{x_3} \left[-x_3^2 + F_2(u_2)\right]$

$F_2(u_2) = \max_{x_2} \left[x_2^2 + F_1(u_1)\right]$

Let $x_1 = u_1 = u_2 - x_2$, $F_1(u_1) = x_1^2 = (u_2 - x_2)^2$

Substitute into $F_2(u_2)$ we get

$$F_2(u_2) = \max_{d_2} \left[x_2^2 + u_2^2 - 2u_2x_2 + x_2^2\right]$$

At the maximum point $(\partial^2 F_2/\partial x^2) = 0$, we obtain $x_2 = 1/2u_2$

Hence

$$F_2(u_2) = \frac{1}{2}u_2^2$$

Now

$$F_3(u_3) = \max_{d_3} \left[-x_3^2 + \frac{1}{2} u_2^2 \right]$$

But

$$u_2 = u_3 - x_3, \quad u_2 = u_3 - x_3, \quad F_3(u_3) = \max_{x_3} \left[\frac{1}{2} x_3^2 + \frac{1}{2} u_3^2 - u_3 x_3 \right]$$

For $(\partial^2 F_2/\partial x_2) = -x_3 - u_3 = 0$, we obtain $x_3 = -u_3$

Hence, $F_3(u_3) = u_3^2$

But $u_3 = x_1 + x_2 + x_3 \geq 14$, thus $u_{3\max} = 14$

$\therefore F_3(u_3)_{\max} = 196 = \max$ value of $(x_1^2 + x_2^2 - x_3^2)$

Back substituting to calculate x_1, x_2 and x_3 gives

$$x_3 = -u_3 = -14$$

$$u_2 = u_3 - x_3 = 28$$

$$x_2 = \frac{1}{2} u_2 = 14$$

$$u_1 = u_2 - x_2 = 14$$

$$x_1 = u_1 = 14$$

SOLUTION

$f_{\max} = 196$, $x_1 = 14$, $x_2 = 14$, and $x_3 = -14$.

8.12 Conclusions

This chapter presented DP as an optimization approach able to transform a complex problem into a sequential set of simpler problems. Both discrete and continuous problems were considered. The formulation of a multistage decision process was explained step by step by discussing its representation as well as the types of multistage decision problems. Characteristics of the DP method were discussed in detail. The concept of suboptimization and the principle of optimality were used as an initial stage in presenting the formulation of DP as a consequence of the simple optimization problem. This chapter also discussed the different forms of DP approaches used in continuous and discrete optimization problems, in particular forward and backward DP. The derivation of the recursive formulae used for both approaches was presented. Computational procedures were shown through some illustrative examples that also explained the computational economy in DP. All the

discussion was devoted to optimization problems with only one constraint; however, a discussion for systems with more than one constraint and the conversion of a final value problem into an initial value problem was presented in the last two sections of the chapter, with an illustrative example provided to support the argument. Finally, the chapter gave a set of unsolved problems for training and understanding the DP approach.

8.13 Problem Set

PROBLEM 8.13.1

Consider a transportation problem with m sources and n destinations. Let a_i be the amount available at source i, $i = 1, 2, \ldots, m$, and let b_j be the amount demanded at destination j, $j = 1, 2, \ldots, n$. Assuming that the cost of the transporting x_{ij} units from source i to destination j is $h_{ij}(x_{ij})$, formulate the problem as a DP model.

PROBLEM 8.13.2

Solve the following linear programming problem as a DP model.

Maximize $Z = 4x_1 + 4x_2$

Subject to

$2x_1 + 7x_2 \leq 21$

$7x_1 + 2x_2 \leq 21$

$x_i \geq 0$

and nonnegative for all is.

PROBLEM 8.13.3

Solve the following nonlinear programming problem as a DP model.

Maximize $Z = 7x_1^2 + 6x_1 + 5x_2^2$

Subject to

$x_1 + 2x_2 \leq 10$

$x_1 - 3x_2 \leq 9$

$x_1 \geq 0$

and nonnegative for all is.

TABLE 8.18

Data for Problem 8.13.4

Resource Unit	Plant 1	Plant 2	Plant 3
1	5	7	8
2	9	10	11
3	12	12	14
4	14	13	16

PROBLEM 8.13.4

Find the allocation for maximal profit when four resource units are available as shown in Table 8.18, where the numbers denote dollars ($) per resource unit.

PROBLEM 8.13.5

Maximize the hydroelectric power $P(X)$, $X=(x_1,x_2,x_3)^T$, produced by building dams on three different river basins, where

$$P(X) = f_1(x_1) + f_2(x_2) + f_3(x_3),$$

and $f_i(x_i)$ is the power generated from the ith basin by investing resource x_i. The total budgetary provision is $x_1 + x_2 + x_3 \leq 3$. The functions f_1, f_2, and f_3 are given in Table 8.19. An integer solution is required for this problem.

PROBLEM 8.13.6

Use Table 8.20 regarding buying and selling only in a deregulated power system with limited energy storage. A utility has a limited electrical energy storage parity of 1000 MW h and must use this to the greatest advantage. The price of the energy changes from month to month in the year. Assume that the energy storage system must be inspected (cleaned) once a year: it takes one month to complete and is scheduled for the first of July each year.

TABLE 8.19

Data for Problem 8.13.5

x_i	1	2	3	4
f_1	0	2	4	6
f_2	0	1	5	6
f_3	0	3	5	6

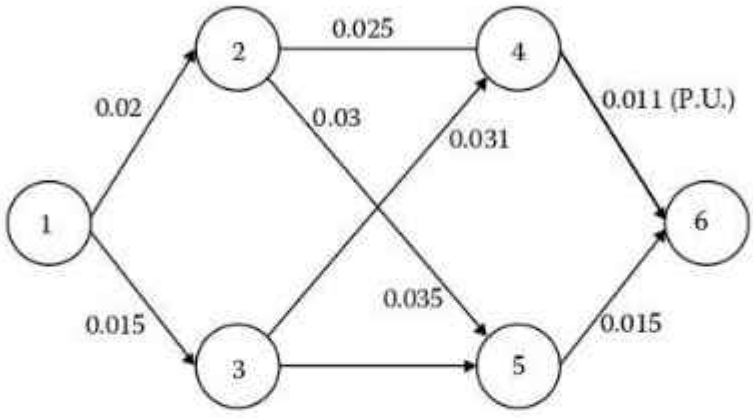
TABLE 8.20

Data for Problem 8.13.6

	Price	
Months	**($)**	**MW h**
January	18	20
February	18	19
March	18	16
April	15	17
May	14	16
June	15	16
July	17	16
August	17	17
September	17	18
October	17	19
November	17	19
December	18	19

PROBLEM 8.13.7

The network shown in Figure 8.11 represents a DC transmission system. In the network, the nodes represent substations. The values associated with the branches are voltage drops (pu). Derive an optimal policy of supplying a load at node (6) from the generation at node (1) with minimum voltage drop.



PROBLEM 8.13.8

(a) Define the dynamic program. Illustrate with all the notations and blocks change for multistage problem.

(b) Showing all steps, use DP method to solve the following problem.

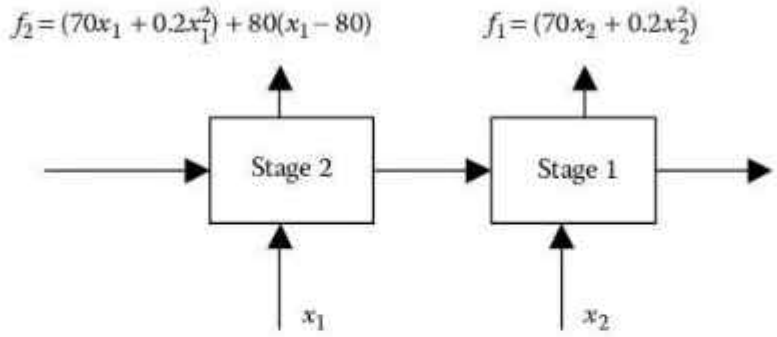


FIGURE 8.11
DC transmission system for Problem 8.13.7.

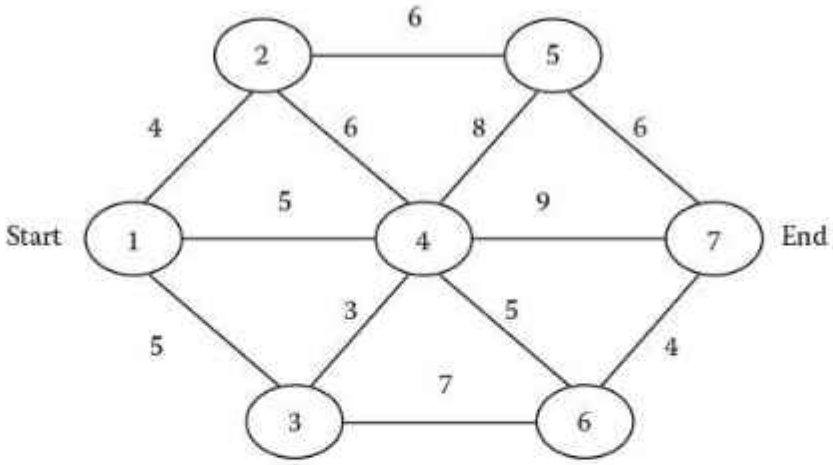
$$\text{Maximize } P = (4d_1 - d_1^2) + (4d_2 - 2d_2^2) + (4d_3 - 3d_3^2) + (4d_4 - 4d_4^2)$$

$$\text{s.t. } d_1 + d_2 + d_3 + d_4 = C$$

$$d_k \geq 0, \quad k = 1, \ldots, 4$$

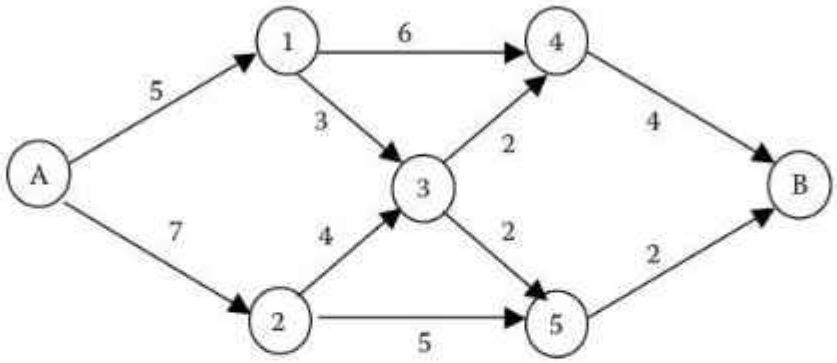
PROBLEM 8.13.9

Use DP technique to find the maximum path through the following network.



PROBLEM 8.13.10

Shortest-route problem: the network below has different routes from A to B by passing through a number of other points. The lengths of the individual routes are shown. Determine the minimum distance from A to B using DP approach. (Find the optimal solution with clear definition of the stages, states, and return function.)



PROBLEM 8.13.11

Solve the following problem by using DP technique

$$\text{Minimize } \sum_{i=1}^{10} y_i^2$$

Subject to

$$\prod_{i=1}^{10} y_i = 8,\ y_i$$

PROBLEM 8.13.12

Solve the following linear programming problem using DP:

$$\text{Maximize } z = 4x_1 + 10x_2$$

Subject to

$$2x_1 + 7x_2 \leq 18$$

$$7x_1 + 2x_2 \leq 14$$

$$x_1, x_2 \geq 0$$

References

1. Bellman, R., *Dynamic Programming*, Princeton University Press, Princeton, NJ, 1957.
2. Bellman, R. and Dreyfus, S., *Applied Dynamic Programming*, Princeton University Press, Princeton, NJ, 1962.
3. Dano, S., *Nonlinear and Dynamic Programming*, Springer-Verlag, New York, 1975.

4. Hadley, G. F., *Nonlinear and Dynamic Programming*, Addison-Wesley, Reading, MA, 1964.
5. Howard, R. A., *Dynamic Programming and Markov Processes*, Wiley, New York, 1960.
6. Kaufman, A. and Cruon, R., *Dynamic Programming*, Academic Press, New York, 1967.
7. Larson, R. E., *State Increment Dynamic Programming*, American Elsevier, New York, 1968.
8. Markowitz, H. and Manne, A. S., On the solution of discrete programming problems, *Econometrica*, 25(1), 84–110, January 1957.
9. Neinhauser, G. L., *Introduction to Dynamic Programming*, Wiley, New York, 1966.
10. Sasieni, M. W., Yaspan, A., and Friedman, A., *Operations Research, Methods and Problems*, Wiley, New York, 1959.
11. Wagner, H. M., *Principles of Management Sciences*, Prentice-Hall, Englewood Cliffs, NJ, 1970.
12. Denardo, E. V., *Dynamic Programming Models and Applications*, Prentice Hall, Englewood Cliffs, NJ, 1982.
13. Dreyfus, S. E. and Law, A. M., *The Art and Theory of Dynamic Programming*, Academic Press, New York, 1977.
14. Dixon, P. and Norman, J. M., An Instructive Exercise in Dynamic Programming, *IIE Transactions*, 16(3), 1984, 292–294.

9

Lagrangian Relaxation

9.1 Introduction

In the last decade, Lagrangian relaxation has grown from a successful but largely theoretical concept to a tool that is the backbone of a number of large-scale applications. Although there have been several surveys of Lagrangian relaxation (e.g., Fisher [1] and Geofrion [2] and an excellent textbook treatment by Muckstadt and Koenig [6]), more extensive use of Lagrangian relaxation in practice has been inhibited by the lack of a "how to do it" exposition similar to the treatment usually extended to linear, dynamic, and integer programming in operations research texts. This chapter is designed to at least partially fill that void and should be of interest to both developers and users of Lagrangian relaxation algorithms.

In many large-scale optimization problems for planning in production, feasibility of the optimization problems present in linear programming (LP) allows the determination of optimal solution by first decomposing a problem into smaller subproblems and then solving the subproblems almost independently. The procedure would allow the procedure of many large-scale problems completely solvable is the decomposition. It exploits the sparsity of many large-scale problems. Several practical applications in various fields such as power systems, management sciences, and economics have taken advantage of Lagrangian relaxation in solving complex scheduling problems [3,6–9].

Many variants of Lagrange relaxation have been posed by researchers in power systems operation for unit commitment (UC). The technique has been developed to take advantage of iterative technique and eliminate certain coupling constraints by adding them to the master problem thereby enabling the independent and simultaneous solutions of many resulting subproblems. Again, in the power system area it is a preferred solution method for mostly the large set of constraints in a UC problem with hydrodispatch problem where start-up, shutdown, and ramp rates are involved. Other approaches such as resistive planning in optimal power flow (OPF) have shown some promise where security constraints of OPF for subproblems are used. This technique allows for speed which hardly offset constraints and easy procedure

exists for relaxing constraints by adding them as part to objective function for optimization.

Lagrangian relaxation is based upon the observation that many difficult programming problems can be modeled as relatively easy problems complicated by a set of side constraints. To exploit this observation, we create a Lagrangian problem in which the complicating constraints are replaced with a penalty term in the objective function involving the amount of violation of the constraints and their dual variables. The Lagrangian problem is easy to solve and provides an upper bound (to a maximization problem) on the optimal value of the original problem. It can thus be used in place of LP relaxation to provide bounds in a branch-and-bound algorithm. The Lagrangian approach offers a number of important advantages over LP relaxations.

The Lagrangian relaxation concept is first formulated in general terms and its use is then demonstrated extensively on a numerical example.

9.2 Concepts

Consider an integer programming problem of the following form:

$$\left.\begin{array}{l} Z = \max[\mathbf{cx}] \\ \text{Subject to} \\ \mathbf{Ax} \leq \mathbf{b} \\ \mathbf{Dx} \leq \mathbf{e} \\ \mathbf{x} \geq 0 \text{ and integral} \end{array}\right\}, \tag{9.1}$$

where

$\mathbf{x}$ is the $n \times 1$ vector and the elements of $\mathbf{x}$ are integers
$\mathbf{b}$ is the $m \times 1$ vector
$\mathbf{e}$ is of order $k \times 1$, and all other matrices have conformable dimension

We assume that the constraints of (9.1) have been partitioned into the two sets $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{Dx} \leq \mathbf{e}$ so that (9.1) is relatively easy to solve if the constraint set $\mathbf{Ax} \leq \mathbf{b}$ is removed. To create the Lagrangian problem, we first define an $\mathbf{m}$ vector of nonnegative multipliers u and add the nonnegative term $u(\mathbf{b} - \mathbf{Ax})$ to the objective function of (9.1) to obtain Equation 9.2:

$$\left.\begin{array}{l} \text{Maximize } \mathbf{cx} + u[\mathbf{b} - \mathbf{Ax}] \\ \text{Subject to} \\ \mathbf{Ax} \leq \mathbf{b} \\ \mathbf{Dx} \leq \mathbf{e} \\ \mathbf{x} \geq 0 \text{ and integral} \end{array}\right\}. \tag{9.2}$$

It is clear that the optimal value of this problem for u fixed at a nonnegative value is an upper bound on Z because we have merely added a nonnegative

TABLE 9.1

Several Issues Related to a Lagrangian Relaxation-Based System

Question	Answer
(a) Which constraints should be relaxed?	The relaxation should make the problem significantly easier, but not too easy
(b) How to compute good multipliers u	There is a choice between a general-purpose procedure called the subgradient method and smarter methods that may be better but which are, however, highly problem specific
(c) How to deduce a good feasible solution to the original problem given a solution to the relaxed problem	The answer tends to be problem specific

term to the ejective function. At this point, we create the Lagrangian problem by removing the constraints $\mathbf{Ax} \leq \mathbf{b}$ to obtain

$$Z_D(u) = \text{Max}[\mathbf{ex}] + u[\mathbf{b} - \mathbf{Ax}]$$

Subject to

$$\mathbf{Dx} < \mathbf{e}$$

$$\mathbf{x} > 0.$$

Since removing the constraints $\mathbf{Ax} \leq \mathbf{b}$ cannot decrease the optimal value, $Z_D(u)$ is also an upper bound on Z. Moreover, by assumption the Lagrangian problem is relatively easy to solve. There are three major questions in designing a Lagrangian relaxation-based system, and some answers can (roughly speaking) be given. Table 9.1 summarizes these issues.

A numerical example is used to illustrate considerations (a), (b), and (c) as well as to compare Lagrangian relaxation to the use of LP to obtain bounds for use in a branch-and-bound algorithm.

9.3 Subgradient Method for Setting the Dual Variables

The subgradient method is demonstrated by an illustrative example. Consider the following problem.

$$\text{Maximize } Z = 16x_1 + 10x_2 + 4x_4 \tag{a}$$

Subject to

$$8x_1 + 2x_2 + x_3 + 4x_4 = 10 \tag{b}$$

$$x_1 + x_2 \leq 1, \tag{c}$$

$$x_3 + x_4 \leq 1. \tag{d}$$

The numerical example is used to develop and demonstrate a method for obtaining dual variable values that produce a tight bound. Ideally, u should solve the following dual problem,

$$Z_D = \text{Min}\, Z_d(u) \quad \text{for } u \geq 0.$$

Before presenting an algorithm for this problem, it is useful to develop some insight by trying different values for the single duel variable u in the example. We use

$$Z_D(u) = \max (16 - 8u)x_1 + (10 - 2u)x_2 + (0 - u)x_3 + (4 - 4u)x_4 + 10u. \quad \text{(e)}$$

Take $u = 0$
Here we have

$$Z_D(u) = \max \{16x_1 + 10x_2 + 0x_3 + 4x_4\}.$$

Note that x_1 has a coefficient larger than that of x_2, and x_4 has a coefficient larger than that of x_3.

$$\begin{aligned} x_1 = 1 \quad &\text{and} \quad x_2 = 0 \quad \text{satisfy } x_1 + x_2 \leq 1 \\ x_3 = 0 \quad &\text{and} \quad x_4 = 0 \quad \text{satisfy } x_3 + x_4 \leq 1. \end{aligned}$$

By substitution we get:

$$Z_D(0) = 20.$$

Also from (a):

$$Z = [16x_1 + 10x_2 + 4x_4] = 20.$$

We now test constraint (b):

$$8x_1 + 2x_2 + x_3 + 4x_4 = 12,$$

which is a violation. Therefore, this solution is not feasible.

It is useful to think of the single constraint (b) that we have dualized as a resource constraint with the right-hand side representing the available supply of some resource and the left-hand side the amount of the resource demanded in a particular solution. We can then interpret the dual variable u as a price charged for the resource. It turns out that if we can discover a price for which the supply and demand for the resource are equal, then this value will also give a tight upper bound. However, such a price might not exist.

With $u = 0$, we discover that the Lagrangian relaxation solution demand for the resource exceeds the available supply by two units, suggesting that we should use a larger value for u.

Take $u = 6$
Here we now have

$$Z_D(6) = \max\{-32x_1 - 2x_2 - 6x_3 - 20x_4 + 60\}.$$

All coefficients are negative, and maximization takes place with all variables set to 0.

$$\begin{aligned} x_1 = 0 \quad &\text{and} \quad x_2 = 0 \quad \text{satisfy } x_1 + x_2 \leq 1 \\ x_3 = 0 \quad &\text{and} \quad x_4 = 0 \quad \text{satisfy } x_3 + x_4 \leq 1. \end{aligned}$$

By substitution we get:

$$Z_D(6) = 60.$$

Also from (a):

$$Z = [16x_1 + 10x_2 + 4x_4] = 0.$$

We now test constraint (b)

$$8x_1 + 2x_2 + x_3 + 4x_4 = 0.$$

This solution is feasible. For $u = 6$ we discover that we have overcorrected in the sense that all variables are 0 in the Lagrangian solution and none of the resource is used. We next try a sequence of dual values in the interval between 0 and 6.

Take $u = 3$
Here we have:

$$Z_D(3) = \max\{-8x_1 + 4x_2 - 3x_3 - 8x_4\} + 30.$$

The variables with negative coefficients are set to 0.

$$\begin{aligned} x_1 = 0 \quad &\text{and} \quad x_2 = 1 \quad \text{satisfy } x_1 + x_2 \leq 1 \\ x_3 = 0 \quad &\text{and} \quad x_4 = 0 \quad \text{satisfy } x_3 + x_4 \leq 1. \end{aligned}$$

By substitution we get:

$$Z_D(3) = 34.$$

Also from (a):

$$Z = [16x_1 + 10x_2 + 4x_4] = 10.$$

We now text constraint (b):

$$8x_1 + 2x_2 + x_3 + 4x_4 = 0.$$

This solution is feasible.

Take $u = 2$
Here we have

$$Z_D(2) = \max\{6x_2 - 2x_3 - 4x_4\} + 20.$$

Thus:

$$Z = [16x_1 + 10x_2 + 4x_4] = 14.$$

$$x_1 = 0 \quad \text{and} \quad x_2 = 1 \quad \text{satisfy } x_1 + x_2 \leq 1$$
$$x_3 = 0 \quad \text{and} \quad x_4 = 0 \quad \text{satisfy } x_3 + x_4 \leq 1.$$

By substitution we get:

$$Z_D(2) = 26.$$

Also from (a):

$$Z = [16x_1 + 10x_2 + 4x_4] = 10.$$

We now test constraint (b):

$$8x_1 + 2x_2 + x_3 + 4x_4 = 2 \leq 10.$$

This solution is feasible.

Take $u = 1$
Here we have

$$Z_D(1) = \max((16 - 8*1)x_1 + (10 - 2*1)x_2 + (0 - 1)x_3 + (4 - 4*1)x_4 + 10*1).$$

Thus

$$Z_D(1) = \max\{8x_1 + 8x_2 - x_3 - 0x_4\} + 10*1.$$

We definitely have $x_3 = 0$, since its coefficient is negative.

Take $x_1 = 1$

$$x_1 = 0 \quad \text{and} \quad x_2 = 1 \quad \text{satisfy } x_1 + x_2 \leq 1$$
$$x_3 = 0 \quad \text{and} \quad x_4 = 0 \quad \text{satisfy } x_3 + x_4 \leq 1.$$

By substitution we get:

$$Z_D(1) = 18.$$

Also from (a):

$$Z = [16x_1 + 10x_2 + 4x_4] = 16.$$

We now test constraint (b):

$$8x_1 + 2x_2 + x_3 + 4x_4 = 8 \leq 10.$$

This solution is feasible. We still have another option:

$$x_3 = 0 \quad \text{and} \quad x_4 = 1 \quad \text{satisfy } x_3 + x_4 \leq 1.$$

By substitution we get:

$$Z_D(1) = 18.$$

Also from (a):

$$Z - [16x_1 + 10x_2 + 4x_4] = 20.$$

We now test constraint (b):

$$8x_1 + 2x_2 + x_3 + 4x_4 = 12.$$

This solution is not feasible.

Take $x_1 = 0$

$$x_1 = 0 \quad \text{requires } x_2 = 1 \quad \text{to satisfy } x_1 + x_2 \leq 1$$
$$x_3 = 0 \quad \text{requires } x_4 = 0 \quad \text{to satisfy } x_3 + x_4 \leq 1.$$

By substitution we get:

$$Z_D(1) = 18.$$

Also from (a):

$$Z = [16x_1 + 10x_2 + 4x_4] = 10.$$

We now test constraint (b):

$$8x_1 + 2x_2 + x_3 + 4x_4 = 2 \leq 10.$$

This solution is feasible. There still exists another option:

$$x_3 = 0 \quad \text{and} \quad x_4 = 1 \quad \text{satisfy } x_3 + x_4 \leq 1.$$

By substitution we get:

$$Z_D(1) = 18.$$

Also from (a):

$$Z = [16x_1 + 10x_2 + 4x_4] = 14.$$

We now test constraint (b):

$$8x_1 + 2x_2 + x_3 + 4x_4 = 6 \leq 10.$$

This solution is feasible.

In the case of $u = 1$, we see that there are four alternative optimal Lagrangian solutions. Table 9.2 gives a list of seven values for u, together with the associated Lagrangian relaxation solution, the bound $Z_D(u)$, and Z for those Lagrangian solutions that are feasible for $u = 2$. For the values tested, the tightest bound of 18 was obtained with $u = 1$, but at the moment we lack any means for confirming that it is optimal.

It is possible to demonstrate that 18 is the optimal value for $Z_D(u)$ by observing that if we substitute any x into the objective function for the Lagrangian problem, we obtain a linear function in u. We use:

$$Z_D(u) = \max (16 - 8u)x_1 + (10 - 2u)x_2 + (0 - u)x_3 + (4 - 4u)x_4 + 10u.$$

TABLE 9.2

List of u Values with the Associated Lagrangian Relaxation Solution, the Bound $Z_D(u)$, and Z for those Lagrangian Solutions

u	x_1	x_2	x_3	x_4	$Z_D(u)$	Value of Lagrangian Solution Z (if Feasible)
0	1	0	0	1	20	—
6	0	0	0	0	60	0
3	0	1	0	0	34	10
2	0	1	0	0	26	10
1	1	0	0	0	18	16
—	1	0	0	1	18	—
—	0	1	0	0	18	10
—	0	1	0	1	18	14
1/2	1	0	0	1	19	—
3/4	1	0	0	1	18.5	—

Take $x_1 = x_2 = x_3 = x_4 = 0$

$$\begin{aligned} Z_D(u) &= \max(16 - 8u)0 + (10 - 2u)0 + (0 - u)0 + (4 - 4u)0 + 10u \\ &= 10u. \end{aligned}$$

Take $x_1 = 1$, $x_2 = x_3 = x_4 = 0$

$$\begin{aligned} Z_D(u) &= \max(16 - 8u)1 + (10 - 2u)0 + (0 - u)0 + (4 - 4u)0 + 10u \\ &= 16 + 2u. \end{aligned}$$

Take $x_2 = 1$, $x_1 = x_3 = x_4 = 0$

$$\begin{aligned} Z_D(u) &= \max(16 - 8u)0 + (10 - 2u)1 + (0 - u)0 + (4 - 4u)0 + 10u \\ &= 10 + 8u. \end{aligned}$$

Take $x_2 = x_4 = 1$, $x_1 = x_3 = 0$

$$\begin{aligned} Z_D(u) &= \max(16 - 8u)0 + (10 - 2u)1 + (0 - u)0 + (4 - 4u)0 + 10u \\ &= 14 + 4u. \end{aligned}$$

Take $x_1 = x_4 = 1$, $x_2 = x_3 = 0$

$$\begin{aligned} Z_D(u) &= \max(16 - 8u)1 + (10 - 2u)0 + (0 - u)0 + (4 - 4u)1 + 10u \\ &= 20 - 2u. \end{aligned}$$

Figure 9.1 exhibits this family of linear functions for all Lagrangian relaxation solutions that are optimal for at least one value of u.

The fact that we must maximize the Lagrangian objective means that for any particular value of u, $Z_D(u)$ is equal to the largest of these linear functions. Thus, the $Z_D(u)$ function is given by the upper envelope of this family of linear equations, shown as a darkened piecewise linear function in Figure 9.1. From this figure it is easy to see that $u = 1$ minimizes $Z_D(u)$.

Figure 9.2 also provides motivation for a general algorithm for finding u. As shown, the $Z_D(u)$ function is convex and differentiate except at points where the Lagrangian problem has multiple optimal solutions. At differentiable points, the derivative of $Z_D(u)$ with respect to u is given by $8x_1 + 2x_2 + x_3 + 4x_4 - 10$, where x is an optimal solution to (LR_u).

These facts also hold in general with the gradient of the $Z_D(u)$ function at differentiable points given by $(\mathbf{Ax} - \mathbf{b})$. These observations suggest that it might be fruitful to apply a gradient method to the minimization of $Z_D(u)$ with some adaptation at the points where $Z_D(u)$ is nondifferentiable. This has been properly achieved in a procedure called the subgradient method. At points where $Z_D(u)$ is nondifferentiable, the subgradient method selects arbitrarily from the set of alternative optimal Lagrangian solutions and uses the vector $(\mathbf{Ax} - \mathbf{b})$ for the solution as though it were the gradient of $Z_D(u)$.

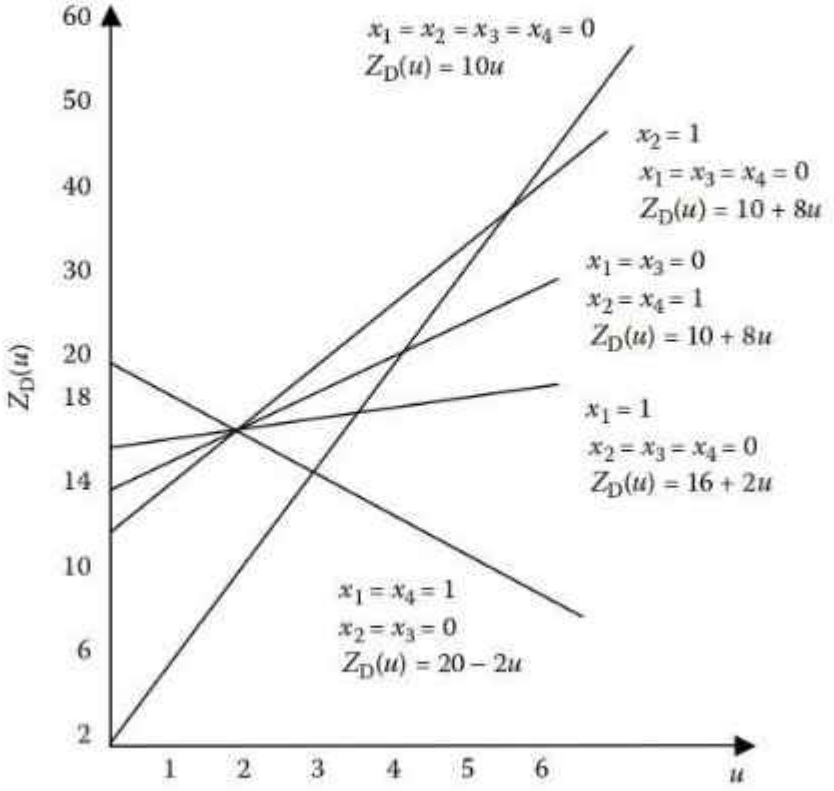


FIGURE 9.1
Family of linear equations for Lagrangian relaxation solution.

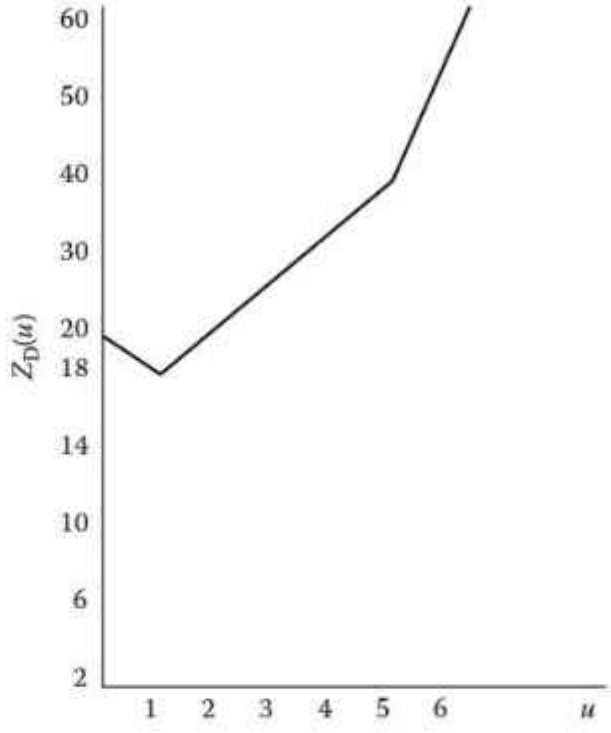


FIGURE 9.2
Composite behavior of $Z_D(u)$ featuring its convex nature and its nondifferentiability property.

The result is a procedure that determines a sequence of values for u by beginning at an initial point u^0 and applying the formula

$$u^{k+1} = \max\{0,\ u^k - t_k(b - Ax^k)\}.$$

In this formula, t_k is a scalar stepsize and x^k is an optimal solution to (LR_d^k), the Lagrangian problem with dual variables set to u^k. The nondifferentiability also requires some variation in the way the stepsize is normally set in a gradient method.

9.4 Setting t_k

To gain insight into a sensible procedure for setting t_k, we discuss the results of the subgradient method applied to the example with three different rules for t_k. For example, we have

$$u^{k+1} = \max\left\{0,\ u^k - t_k\left(10 - 8x_1^{(k)} - 2x_2^{(k)} - 4x_4^{(k)}\right)\right\} \tag{9.3}$$

9.4.1 Case 1: Subgradient Method with $t_k = 1$ for All k

In this first case, t_k is fixed at 1 on all iterations. As a result the formula is

$$u^{k+1} = \max\left\{0,\ u^k - \left(10 - 8x_1^{(k)} - 2x_2^{(k)} - 4x_4^{(k)}\right)\right\}.$$

We start with

$$u^0 = 0.$$

Recall the table entry for $u = 0$.

u	x_1	x_2	x_3	x_4
0	1	0	0	1

$$\begin{aligned} u^1 &= \max\{0,\ 0 - (10 - 8 \times 1 - 2 \times 0 - 4 \times 1)\} \\ &= \max\{0,\ -(-2)\} \\ u^1 &= 2. \end{aligned}$$

Recall the table entry for $u = 2$.

u	x_1	x_2	x_3	x_4
2	0	1	0	0

$$\begin{aligned} u^2 &= \max\{0,\ 2 - (10 - 8 \times 0 - 2 \times 1 - 4 \times 0)\} \\ &= \max\{0,\ -6\} \\ u^2 &= 0. \end{aligned}$$

We now use values for $u = 0$ as before,

$$u^3 = 0 - (-2) = 2,$$

and use values for $u = 2$ as before,

$$u^4 = \max[0, 2 - 8] = 0.$$

We see that the subgradient method oscillates between the values $u = 0$ and $u = 2$.

9.4.2 Case 2: Subgradient Method with $t_k = 1, 0.5, 0.25, \ldots$

In this second case, t_k converges to 0 with each successive value equal to half the value of the previous iteration.

$$t_k = \left(\frac{1}{2}\right)^k.$$

Therefore,

$$u^{k+1} = \max\left\{0, u^k - \left[\frac{1}{2}\right]^k \left(10 - 8x_1^{(k)} - 2x_2^{(k)} - 4x_4^{(k)}\right)\right\}.$$

We start once again with

$$u^0 = 0.$$

Here we have $t_0 = 1$, and we get the same result as in the preceding case.

$$u^1 = 0 - (-2) = 2.$$

With $u = 2$, we get

u	x_1	x_2	x_3	x_4
2	0	1	0	0

$$\begin{aligned} u^2 &= \max\{0, u^{12} - [1/2]^1(10 - 8 \times 0 - 2 \times 1 - 4 \times 0)\} \\ &= \max\{0, 2 - 0.5(10 - 2)\} \\ &= \max\{0, 2 - 4\} \\ u^2 &= 0. \end{aligned}$$

Proceeding similarly, we get

$$u^3 = 0 - \left(\frac{1}{4}\right)(-2) = \frac{1}{2}$$

$$u^4 = \frac{1}{2} - \left(\frac{1}{8}\right)(-2) = \frac{3}{4}$$

$$u^5 = \frac{3}{4} - \left(\frac{1}{16}\right)(-2) = \frac{7}{8}$$

$$u^6 = \frac{7}{8} - \left(\frac{1}{32}\right)(-2) = \frac{15}{16}.$$

In this case, the subgradient method behaves nicely and converges to the optimal value of u.

9.4.3 Case 3: Subgradient Method with $t_k = 1, 1/3, 1/9, \ldots$

In this final case, t_k also converges to 0, but more quickly. Each successive value is equal to one-third the value of the previous iteration,

$$t_k = \left(\frac{1}{3}\right)^k.$$

Therefore,

$$u^{k+1} = \max\left\{0,\ u^k - \left[\frac{1}{3}\right]^k \left(10 - 8x_1^{(k)} - 2x_2^{(k)} - 4x_4^{(k)}\right)\right\}.$$

We start once again with

$$u^0 = 0$$

$$u^1 = 0 - (-2) = 2$$

$$u^2 = \max\left(0, 2 - \left(\frac{1}{3}\right)(8)\right) = 0$$

$$u^3 = 0 - \left(\frac{1}{9}\right)(-2) - \frac{2}{9}$$

$$u^4 = \frac{2}{9} - \left(\frac{1}{27}\right)(-2) = 0.296$$

$$u^5 = 0.296 - \left(\frac{1}{81}\right)(-2) = 0.321$$

$$u^6 = 0.321 - \left(\frac{1}{243}\right)(-2) = 0.329$$

$$u^7 = 0.329 - \left(\frac{1}{729}\right)(-2) = 0.332.$$

In this case, the subgradient method converges to $u = 1/3$, showing that if the stepsize converges to 0 too quickly, then the subgradient method will converge to a point other than the optimal solution.

From these examples, we suspect that the stepsize in the subgradient method should converge to 0 but not too quickly. These observations have been confirmed in a result (see Held et al. [5]) that states that if as $k \to \infty$,

$$t_k \to 0 \quad \text{and} \quad \sum_{i=1}^{k} t_i \to \infty$$

then $Z_D(u^k)$ converges to its optimal value Z_D. Note that Case 3 actually violates the second condition since $\sum_{i=1}^{k} t_i \to 2$, thus showing that these conditions are sufficient but not necessary.

A formula for t_k that has proven effective in practice is

$$t_k = \frac{\lambda_k \left[Z_D(u^k) - Z^* \right]}{\sum_{i=1}^{m} \left[b_i - \sum_{j=1}^{n} a_{ij} x_j^k \right]^2}. \tag{9.4}$$

In this formula, Z^* is the objective value of the best-known feasible solution to (p) and λ_k is a scalar chosen between 0 and 2. Frequently the sequence λ_k is determined by starting with $\lambda_k = 2$ and reducing λ_k by a factor of 2 whenever $Z_D(u^k)$ has failed to decrease in a specified number of iterations.

Justification for this formula, as well as many other interesting results on the subgradient method, is given in Held et al. [5]. The feasible value Z^* initially can be set to 0 and then updated using the solutions that are obtained on those iterations in which the Lagrangian problem solution turns out to be feasible in the original problem. Unless we obtain a u^k for which $Z_D(u^k) = Z^*$, there is no way of proving optimality in the subgradient method. To resolve this difficulty, the method is usually terminated upon reaching a specified iteration limit.

Other procedures that have been used for setting multipliers are called multiplier-adjustment methods. Multiplier-adjustment methods are heuristics for the dual problem that are developed for a specific application and exploit some special structures of the dual problem in that approach. The first highly successful example of a multiplier-adjustment method was Erlenkotter's [X] algorithm for the uncapacitated location problem.

By developing a multiplier-adjustment method specifically tailored for some problem class, one is usually able to improve on the subgradient method. However, because the subgradient method is easy to program and has performed robustly in a wide variety of applications, it is usually at least the initial choice for setting the multipliers in Lagrangian relaxation.

Returning to our example, we have obtained through the application of Lagrangian relaxation and the subgradient method a feasible solution with a

value of 16 and an upper bound on the optimal value of 18. At this point, we could stop and be content with a feasible solution proven to be within about 12% of optimality, or we could complete the solution of the example to optimality using branch-and-bound, with bounds provided by our Lagrangian relaxation. In the next section, we show how such an approach compares with more traditional LP-based branch-and-bound algorithms.

9.5 Comparison with LP-Based Bounds

In this section, we compare Lagrangian relaxation with the upper bound obtained by relaxing the integrality requirement on x and solving the resulting linear program. Let Z_{LP} denote the optimal value of (p) with integrality on x relaxed. We start by comparing Z_{LP} for the example with the best upper bound of 18 obtained previously with Lagrangian relaxation. To facilitate this comparison, we first write out the standard LP dual of the example. Let u, v_1, and v_2 denote dual variables on constraints 9.5 through 9.8 and w_j a dual variable on the constraint $x_j \leq 1$.

Then, the LP dual form is

$$\text{Min } 10u + v_1 + v_2 + w_1 + w_2 + w_3 + w_4 \tag{9.5}$$

$$8u + v_1 + w_1 \geq 16 \tag{9.6}$$

$$2u + v_1 + w_2 \geq 10 \tag{9.7}$$

$$u + v_2 + w_3 \geq 0 \tag{9.8}$$

$$4u + v_2 + w_4 \geq 4 \tag{9.9}$$

$$u, v_1, v_2, w_1, \ldots, w_4 \geq 0. \tag{9.10}$$

The optimal solution to the primal LP is $x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 1/2$, and the optimal solution to the dual LP is $u = 1, v_1 = 8, v_2 = w_1 = \cdots = w_4 = 0$. In order to verify that each of these solutions is optimal, we simply substitute them in the primal and dual and observe that each is feasible and gives the same objective value 18.

This exercise has demonstrated two interesting facts: $Z_{LP} = 18$, the same upper bound we obtained with Lagrangian relaxation; and the LP dual variable value of $u = 1$ on constraint 9.7 is exactly the value that gave the minimum upper bound of 18 on the Lagrangian problem. These observations are part of a pattern that holds generally and is nicely summarized in a result from Geoffrion [2] which states that $Z_D \leq Z_{LP}$ for any Lagrangian relaxation. This fact is established by the following sequence of relations between optimization problems.

$$Z_D = \text{Min}\{\text{Max}(\mathbf{cx} + u(\mathbf{b} - \mathbf{Ax}))\}$$
$$u \geq 0$$
$$\mathbf{Dx} \leq \mathbf{e}$$
$$\mathbf{x} \geq 0 \text{ and integral,}$$

by LP duality

$$\text{Max } \mathbf{cx}$$
$$\mathbf{Ax} \leq \mathbf{b}$$
$$\mathbf{Dx} \leq \mathbf{e}$$
$$\mathbf{x} \geq 0.$$

Besides showing that $Z_D \leq Z_{LP}$ the preceding logic indicates when $Z_D = Z_{LP}$ and when $Z_D < Z_{LP}$. The inequality in the sequence of relations connecting Z_D and Z_{LP} is between the Lagrangian problem and the Lagrangian problem with integrality relaxed. Hence, we can have $Z_D < Z_{LP}$ only if this inequality holds strictly or, conversely, $Z_D = Z_{LP}$ only if the Lagrangian problem is unaffected by removing the integrality requirement on x. In the Lagrangian problem for the original example, the optimal values of the variables will be an integer whether we require it or not. This implies that we must have $Z_D = Z_{LP}$, something that we have already observed numerically.

This result also shows that we can improve the upper bound by using a Lagrangian relaxation in which the variables are not naturally integral.

9.6 Improved Relaxation

An alternative relaxation for the example is given below.

$$Z_D(v_1,v_2) = \text{Max}(16 - v_1)x_1 + (10 - v_1)x_2 + (0 - v_2)x_3 + (4 - v_2)x_4 + v_1 + v_2 \tag{9.10}$$

Subject to

$$8x_1 + 2x_2 + x_3 + 4x_4 \leq 10 \tag{9.11}$$
$$0 < x_j < 1, \quad j = 1, \ldots, 4 \tag{9.12}$$
$$x_j \text{ integer}, \quad j = 1, \ldots, 4. \tag{9.13}$$

In this relaxation, we have dualized constraints, Equations 9.12 and 9.13, and obtained a relaxation that is a knapsack problem. Although this problem is known to be difficult in the worst case, it can be solved practically using a variety of efficient knapsack algorithms such as dynamic programming.

TABLE 9.3

Subgradient Method Applied to Improved Relaxation

v_1	v_2	λ_k	x_1	x_2	x_3	x_4	$Z_D(v_1,v_2)$
Feasible with $Z=4$							
0	0	1	1	1	0	0	26
13	0	1	0	0	0	1	17
Feasible with $Z=16$							
0	0	1	1	1	0	0	26
11	0	1	1	0	0	0	16

Because the continuous and integer solutions to the knapsack problem can differ, the analytic result obtained in the previous section tells us that this relaxation may provide bounds that are better than LP.

This is confirmed empirically in Table 9.3, which shows the application of the subgradient method to this relaxation. We begin with both dual variables equal to 0 and in four iterations, we converge to a dual solution in which the upper bound of 16 is equal to the objective value of the feasible solution obtained when we solve the Lagrangian problem. Hence, Lagrangian relaxation has completely solved the original problem. In this example, we have set the stepsize using the formula given previously with $\lambda_k=1$.

This example illustrates that with careful choice of which constraints to dualize, Lagrangian relaxation can provide results that are significantly superior to LP-based branch-and-bound. The choice of constraints is to some extent an art much like the formulation itself. Typically, one will construct several alternative relaxations and evaluate them, both empirically and analytically, using the result on the quality of bounds presented in the previous section. The alternative relaxations can be constructed in one of the two ways: begin with an integer programming formulation and select different constraints to dualize, or alternatively, begin with some easy to solve model such as the knapsack or shortest route problem which is close to, but not exactly the same as, the problem one wishes to solve. Then try to add a set of side constraints to represent those aspects of the problem of interest that are missing in the simpler model. A Lagrangian relaxation can be obtained by dualizing the side constraints that have been added.

9.7 Summary of Concepts

Up to this point, the concept of Lagrangian relaxation has been developed piecemeal on an example. We can now formulate and present a generic Lagrangian relaxation algorithm. Figure 9.3 shows a generic Lagrangian

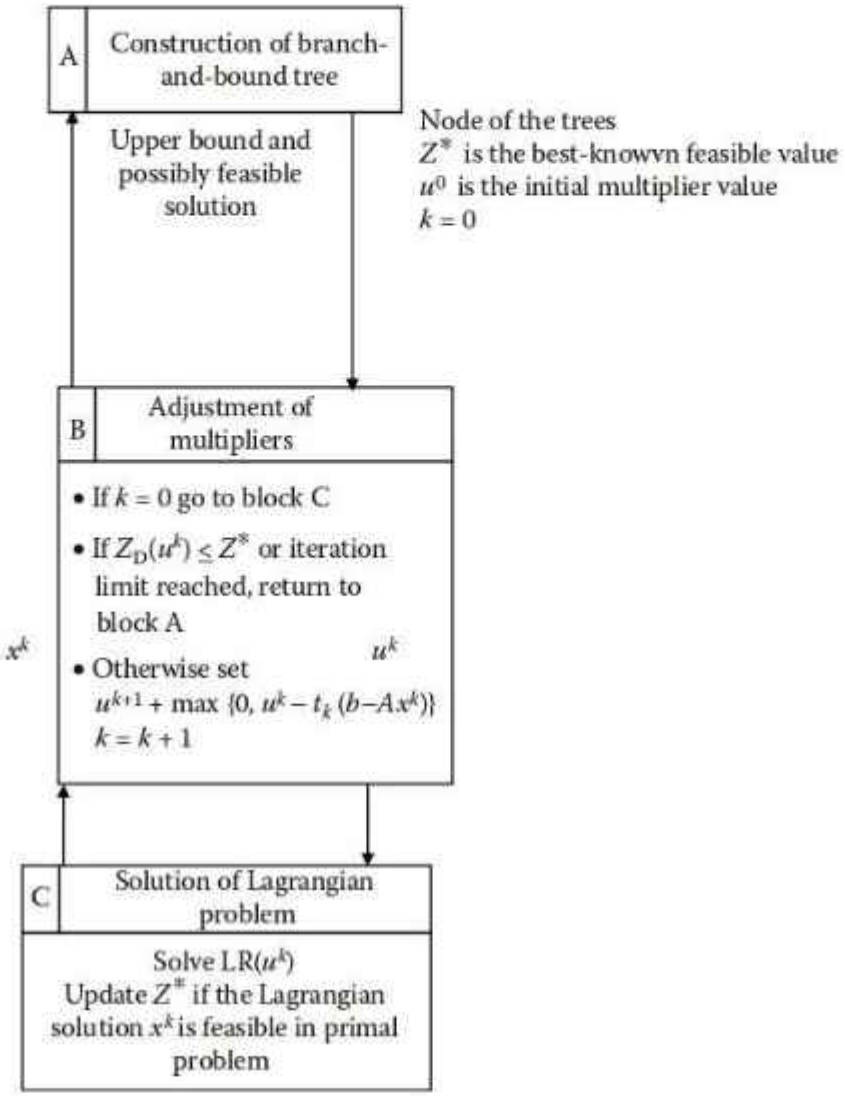


FIGURE 9.3
Generic Lagrangian relaxation algorithm.

relaxation algorithm consisting of several major steps. The first step is the standard branch-and-bound process in which a tree of solution alternatives is constructed with certain variables fixed to specified values at each node of the tree. These specified values are passed from block A to block B together with Z^*, the objective value of the currently best-known feasible solution, and starting multipliers u^0.

In blocks B and C, we iterate between adjusting the multipliers with the subgradient, updating the formula to obtain a new multiplier value u^k, and solving the Lagrangian problem to obtain a new Lagrangian solution x^k. The process continues until we either reach an iteration limit or discover an upper bound for this node that is less than or equal to the current best-known feasible value Z^*. At this point, we pass back to block A the best upper bound we have discovered together with any feasible solution that

may have been obtained as a result of solving the Lagrangian problem. In Fisher's experience, it is rare in practice that the Lagrangian solution will be feasible in the original problem (p). However, it is not uncommon that the Lagrangian solution will be nearly feasible and can be made feasible with some minor modifications. A systematic procedure for doing this can be applied in block C and constitutes what might be called a "Lagrangian heuristic." Lagrangian heuristics have been vital to the computational success of many applications, such as those described in Fisher [1], and may well prove to be as important as the use of Lagrangians to obtain upper bounds.

It is not uncommon in large-scale applications to terminate the process depicted in block B before the branch-and-bound tree has been explored sufficiently to prove optimality. In this case, the Lagrangian algorithm is really a heuristic with some nice properties, such as an upper bound on the amount by which the heuristic solution deviates from optimality.

9.8 Past Applications

A brief description of several instances in which Lagrangian relaxation has been used in practice should give the flavor of the kinds of problems for which Lagrangian relaxation has been successful.

Bell et al. describe the successful application of the algorithm to Air Products and Chemicals, which has resulted in a reduction of distribution cost of about $2 million per year.

Fisher [1] discusses the application, in the Clinical Systems Division of Du Pont, of an algorithm for vehicle routing that is based on a Lagrangian relaxation algorithm for the generalized assignment problem.

Graves and Lamar [4] treat the problem of designing an assembly by choosing from available technology a group of resources to perform certain operations. The choices cover people, single-purpose machines, narrow-purpose pickplace robots, and general-purpose robots. Their work has been applied in a number of industries, including the design of robot assembly systems for the production of automobile alternators. Graves [3] has also discussed the use of Lagrangian relaxation to address production planning problems from an hierarchical perspective.

Mulvey [9] is concerned with condensing a large database by selecting a subset of representative elements. He has developed a Lagrangian relaxation-based clustering algorithm that determines a representative subset for which the loss in information is minimized in a well-defined sense. He has used this algorithm to reduce the 1977 U.S. Statistics of Income File for Individuals maintained by the Office of Tax Analysis from 155,212 records to 74,762 records.

The application described in Shepardson and Marsten [7] involves the scheduling of personnel who must work two duty periods, a morning shift and an afternoon shift. Their algorithm determines optimal schedules for each worker so as to minimize cost and satisfy staffing requirements. Helsinki City Transport has applied this algorithm to bus crew scheduling.

Van Roy and Gelders discuss the use of Lagrangian relaxation for a particular problem arising in distribution.

In each of the applications described above, development of the Lagrangian relaxation algorithm required a level of involvement on the part of skilled analysts that is similar to that required in the use of dynamic programming. Just as some insight into a problem is required before dynamic programming can be applied fruitfully, it is generally nontrivial to discover a Lagrangian relaxation that is computationally effective. Moreover, once this has been done, the various steps in the algorithm must be programmed more or less from scratch. Often this process can be made easier by the availability of an "off the shelf" algorithm for the Lagrangian problem if it is a well-known model, such as a network flow, shortest route, minimum spanning tree, or knapsack problem.

Despite the level of effort required in implementing Lagrangian relaxation, the concept is growing, in popularity because the ability it affords to exploit special problem structure often is the only hope for coping with large real problems.

For the future, it remains to be seen whether Lagrangian relaxation will continue to exist as a technique that requires a significant ad hoc development effort or whether the essential building blocks of Lagrangian relaxation will find their way into user-friendly mathematical programming codes such as LINDO or IFPS OPTIMUM.

9.9 Summary

9.9.1 Overview

The branch-and-bound technique for solving integer programming problems is a powerful solution technique despite the computational requirements involved.

Most computer codes based on the branch-and-bound technique differ from standard known procedures in the details of selecting the branching variable at a node and the sequence in which the subproblems are examined.

These rules are based on heuristics developed through the basic disadvantage of the branch-and-bound algorithm, which is the necessity of solving a complete linear program at each node. In large problems, this could be very time consuming, particularly when the only information needed at the node may be its optimum objective value. This point is clarified by realizing that

once a good bound is obtained, many nodes can be discarded as their optimum objective values are then known.

The preceding point led to the development of a procedure whereby it may be unnecessary to solve all the subproblems of the branch-and-bound tree. The idea is to estimate an upper bound and assume a maximization problem on the optimum objective valued at each node. Should this upper bound become smaller than the objective associated with the best available integer solutions, the node is discarded.

9.9.2 Algorithm of Solution Using Lagrangian Relaxation Approach

Consider an integer programming problem of the following form:

$$Z = \max \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

Subject to

$$\mathbf{Ax} \leq \mathbf{b}$$

$$\mathbf{Dx} \leq \mathbf{e}$$

$$\mathbf{x} \leq 0 \text{ and integer,}$$

where $\mathbf{x}$ is an n*1, elements of $\mathbf{x}$ are integers, $\mathbf{b}$ is an m*1, $\mathbf{c}$ is a k*1, and all other matrices have comformable dimensions. The algorithm is summarized as follows:

Step 1. We assume that the constraints of the problems have been partitioned into two sets $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{Dx} \leq \mathbf{e}$ so that the problem is relatively easy to solve if the constraint set $\mathbf{Ax} \leq \mathbf{b}$ is removed.

Step 2. To create the Lagrangian problem we first define an $\mathbf{m}$ vector of nonegative multipliers U and add the nonnegative term $U(\mathbf{b} - \mathbf{Ax})$ to the objective function to form $Z_{\mathrm{D}}(U)$.

$$Z_{\mathrm{D}}(U) = \max \mathbf{c}^{\mathrm{T}}\mathbf{x} + U(\mathbf{b} - \mathbf{Ax})$$

Subject to

$$\mathbf{Dx} \leq \mathbf{e}$$

$$\mathbf{x} \geq 0 \text{ and integral.}$$

It is clear that the optimal value of this problem for U fixed at a nonnegative value is an upper bound on Z because we have merely added a nonnegative form to the objective function.

Step 3. The new objective $Z_{\mathrm{D}}(U)$ is nondifferentiable at points of the Lagrangian problem solution. This led to the use of the subgradient method for minimization of $Z_{\mathrm{D}}(U)$ with some adaptation at the points where $Z_{\mathrm{D}}(U)$ is nondifferentiable.

Assuming we begin at a multiple value U^0, then

$$U^{k+1} = \max\{0, U^k - t_k(\mathbf{b} - \mathbf{A}x^k)\},$$

t_k is a stepsize, and x^k is an optimal solution to LR$_v k$.

The stepsize, t_k, should converge to 0 but not too quickly; that is,

$$k \to \infty, \quad t_k \to 0, \quad \sum_{i=1}^{k} t_i \to \infty.$$

$Z_D(U^k)$ converges to its optimal value.

A formula for t_k is given by

$$t_k = \frac{\lambda_k[Z_D(U^k) - Z^*]}{\sum_{i=1}^{n}\left[b_i - \sum_{j=1}^{m} a_{ij}X_j^k\right]^2}. \tag{9.14}$$

Z^* is the objective value of the best-known feasible solution and λ_k is a scalar chosen between 0 and 2, and reducing λ_k by a factor of 2 whenever $Z_D(U^k)$ has failed to decrease in a specified number of iterations.

The feasible value Z^* initially can be set to (0) and then updated using the solutions obtained on those iterations in which the LR solution must be feasible in the original problem.

Step 4. Continue the procedure in step 3 until we either reach an iteration limit or discover an upper bound for this node that is less than or equal to the best-known feasible value Z^*.

9.9.3 Power System Application: Scheduling in Power Generation Systems

9.9.3.1 Model

A simplified version of the power scheduling problem is formulated as a mixed integer programming problem having a special structure that facilitates rapid computation.

First, reserve and demand constraints are included in the basic model. Then other constraints are included in the formulation without affecting the basic structure of the problem.

The integer variables in the model indicate whether a specified generating unit is operating during a period.

$x_{it} = 1 \rightarrow$ generating unit i is operating in period t

$x_{it} = 0 \rightarrow$ generating unit i is off

$i = 1, \ldots, I, \quad t = 1, 2, \ldots, T.$

The continuous variable y_{ikt} represents the proportion of the available capacity M_{ik} that is actually used at period t: $k=1,\ldots,k_i$.

$k_i \rightarrow$ number of linear segments of the production cost curve
$yk_{it} \rightarrow 0$ only if $x_{it}=1$; hence this constraint is introduced: $0 \leq y_{ik} \leq x_{it}$.

The total energy output from generator i at time t is given by

$$m_i x_i t + \sum_{k=i}^{k_i} M_{ik} y_{ikt}, \tag{9.15}$$

where
m_i is the minimum unit capacity
M_i is the maximum unit capacity

$$w_i = \begin{cases} 1 & \text{if the unit is started up at time } t \\ 0 & \text{otherwise} \end{cases}$$
$$z_{it} = \begin{cases} 1 & \text{if the unit is shut down in period } t \\ 0 & \text{otherwise} \end{cases}$$

c_i is the start-up cost of generator i
g_i is the operating cost of unit i at its minimum capacity for 1 h
h_t is the number of hours in period t

The objective function to be minimized for the basic T-period scheduling model is

$$\sum_{i=1}^{I} \left\{ \sum_{t=1}^{T} \left\{ c_i w_i + d_i z_{it} + h_i g_i x_{it} + \sum_{k=1}^{k_i} M_{ik} h_t g_{ik} y_{ikt} \right\} \right\}. \tag{9.16}$$

Let D_t represent the demand level at time t. Then, the demand constraint is given by

$$m_i x_{it} + \sum_{k=1}^{k_i} M_{ik} y_{ikt} \geq D_t. \tag{9.17}$$

Let the reserve R_t be the minimum quantity at time t. Then, the reserve constraint is given by

$$\sum_{i=1}^{I} M_i x_{it} \geq R_t. \tag{9.18}$$

By imposing only the constraints

$$w_{it} \geq x_{it} - x_{i,t-1}, \quad w_{it} \geq 0$$
$$z_{it} \geq x_{i,t-1} - x_{it}, \quad z_{it} \geq 0$$
$$x_{it} = 0 \text{ or } 1.$$

then the objective function to be minimized w_{it}, z_{it} must equal either 0 or 1.

The mixed integer model for the basic power scheduling problem is

$$\text{Minimize} \sum_{i=1}^{I} \left\{ \sum_{t=1}^{T} \left\{ c_i w_{it} + d_i z_{it} + h_t g_i x_{it} + \sum_{k=1}^{k_i} M_{ik} h_i g_{ik} y_{ikt} \right\} \right\}. \tag{9.19}$$

Subject to

$$\sum_{i}^{k_i} k = 1 \; m_i x_{it} + \sum_{k=1}^{k_i} M_{ik} y_{ikt} \geq D_t \tag{9.20}$$

$$\sum_{i=1}^{I} M_i x_{it} \geq R_t \tag{9.21}$$

$$0 < y_{ikt} \leq x_{it} \tag{9.22}$$

$$w_{it} \geq x_{it} - x_{i,t-1} \tag{9.23}$$

$$z_{it} \geq x_{i,t-1} - x_{it} \tag{9.24}$$

$$w_{it} \geq 0 \tag{9.25}$$

$$z_{it} \geq 0 \tag{9.26}$$

$$0 \leq x_{it} \leq 1 \tag{9.27}$$

x_{it} integer.

9.9.3.2 *Relaxation and Decomposition of the Model*

Lagrangian relaxation is used to decompose the problem into I single generator subproblems. The advantage of decomposing by generator is that the constraints and costs that depend on the state of the generator from period to period can easily be considered in the subproblems.

The solution of the relaxed problem provides a lower bound on the optimal solution of the original problem. The Lagrangian relaxation model is

$$\sum_{i=1}^{I} \left\{ \sum_{t=1}^{T} \left\{ c_i w_{it} + d_i Z_{it} + h_t g_i x_{it} + \sum_{k=1}^{k_i} M_{ik} h_{it} g_{ik} y_{ikt} \right\} \right\}$$
$$+ \sum_{t=1}^{T} v_t \left(D_t - \sum_{i=1}^{I} m_i x_{it} - \sum_{i=1}^{I} \sum_{k=1}^{k_i} m_{ik} y_{ikt} \right) + \sum_{t=1}^{T} u_t \left(k_t - \sum_{i=1}^{I} m_i x_{it} \right) \tag{9.28}$$

$$C_{\text{set}}\begin{cases} 0 \le y_{ikt} \le x_{it} \\ w_{it} \ge x_{it} - x_{i,t-1} \\ z_{it} \ge x_{i,t-1} - x_{it} \\ w_{it} \ge 0 \\ z_{it} \ge 0 \\ 0 \le x_{it} \le 1 \\ x_{it} \text{ integers} \end{cases}, \tag{9.29}$$

where v_t and u_t are nonnegative real numbers (Lagrange multipliers).

The Lagrangian relaxation problem decomposes into I single generator subproblems of the form:

$$\text{Minimize} \sum_{t=1}^{T} \left\{ c_i w_{it} + d_i z_{it} + (h_t g_i - v_t M_i - v_t m_i) x_{it} + \sum_{k=1}^{k_i} M_{ik}(h_t g_{ik} - v_t) y_{ikt} \right\} \tag{9.30}$$

Subject to

C_{set} of constraints.

Figure 9.4 shows a graph that could be used for the solution of the subproblems in the basic model.

The upper state in each period represents the ON state and the lower state represents the OFF state for the generator. The transition arcs on the graph represent feasible decisions and the arc lengths are the costs associated with the decision. If the generator is OFF in period $t-1$ then $x_{i,t-1}=0$, and if it is ON in period t, $x_{it}=1$.

Consequently, $w_{it} \ge 1$, $z_{it} \ge 0$, $0 \le y_{ikt} \le 1$ to minimize the cost. Given the values of $x_{i,t-1}$ and x_{it} we set $w_{it}=1$, $z_{it}=0$, $y_{ikt}=0$ if $g_{ik}h_t - v_t \ge 0$ and $y_{ikt}=1$ if $g_{ik}H_t - v_t < 0$. The lengths of other arcs can be determined in a similar way.

A path through the graph specifies an operating schedule for the generator and the problem of finding the minimum cost schedule becomes a shortest path problem on a cyclic state graph.

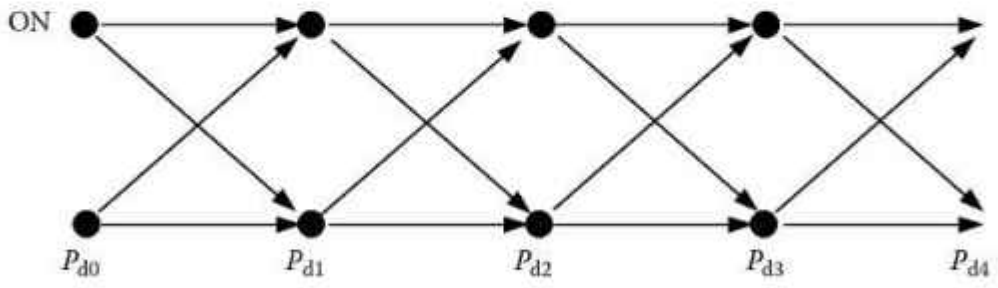


FIGURE 9.4
Basic model state graph.

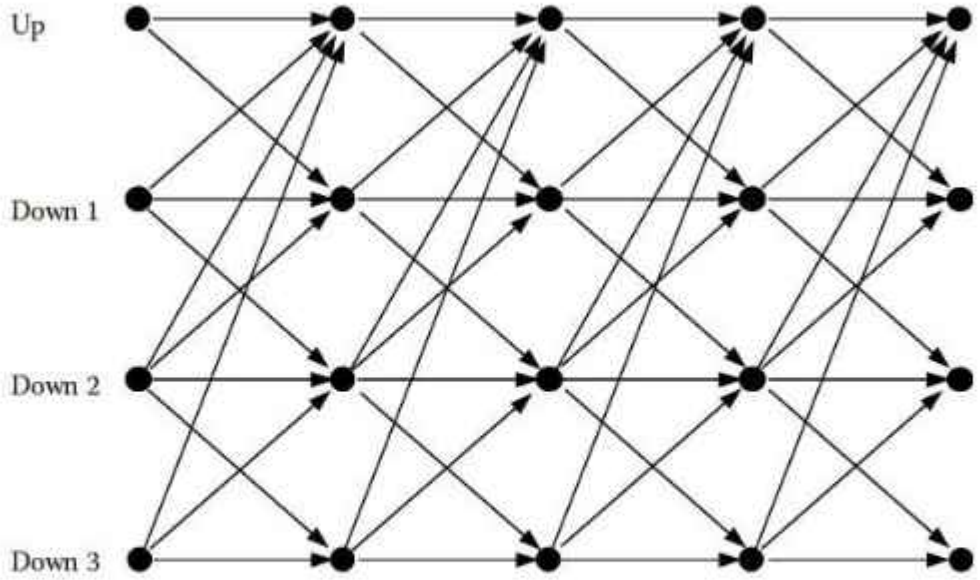


FIGURE 9.5
Graph with time-dependent start-up.

By expanding the state graph, it is also possible to represent certain aspects of the real problem that were omitted from the basic model. The state graph of the model is shown in Figure 9.5.

The state model in Figure 9.5 can accommodate many extensions such as

1. Time-dependent start-up cost
2. Minimum up and down constraints
3. Generator availability restriction

The time-dependent start-up costs are added to the model by varying the costs on the transition arcs that lead from a downstate to an upstate. The minimum up- and downtimes are enforced by eliminating some of the transition arcs in the state graph.

For example, if a generator must be OFF for at least three time periods, the transition arcs from the downstate 1 tend down two states to the up-states in the figure. Two would be eliminated.

9.9.3.3 Solution Technique

The backbone of the technique is a branch-and-bound procedure that builds an enumeration tree for the 0–1 variable x_{ij}. Each node in the tree is characterized by a set of x_{it} variables with fixed values. For the basic model the problem represented by a node in the enumeration tree has the form of the basic problem with the appropriate set of the x_{it} variables fixed. The Lagrangian relaxation of the problem at each node is solved to obtain a lower bound on the optimal solution for the problem at the node. To obtain the solution at a node a simple shortest path algorithm is used at the node to solve each of the I single generator subproblems using dynamic programming.

9.10 Illustrative Examples

Example 9.1

Consider the multidimensional problem

$$\text{Maximize } z = 10x_1 + 5x_2 + 8x_3 + 7x_4$$

Subject to

$$6x_1 + 5x_2 + 4x_3 + 6x_4 \leq 40$$

$$3x_1 + x_2 \leq 15$$

$$x_1 + x_2 \leq 10$$

$$x_3 + 2x_4 \leq 10$$

$$x_j \leq 0.$$

1. Develop the of the Lagrange relaxation formulation for this problem.
2. Formulate the maximization of the subproblem for $k = 0$.

Solution

Step 1. In matrix notation, the problem is

$$\text{Maximize } Z = \mathbf{cx}$$

Subject to

$$\mathbf{Ax} \leq \mathbf{b}$$

$$\mathbf{Dx} \leq \mathbf{e},$$

where $\mathbf{x} = [x_1, x_2, x_3, x_4]^t$ and $\mathbf{c} = [10, 5, 8, 7]^t$, with

$$\begin{bmatrix} 6 & 5 & 4 & 6 \\ 3 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leq \begin{bmatrix} 40 \\ 15 \\ 10 \\ 10 \end{bmatrix}.$$

By partitioning this set of constraints, we obtain

$$\mathbf{Ax} \leq \mathbf{b}$$

$$\mathbf{Dx} \leq \mathbf{e},$$

where

$$\mathbf{A} = [6 \quad 5 \quad 4 \quad 6], \quad \mathbf{b} = [40]$$

$$\mathbf{D} = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 15 \\ 10 \\ 10 \end{bmatrix}.$$

Step 2. Lagrange is defined as

$$Z_D(u) = \text{Max}(\mathbf{cx}) + u(\mathbf{b} - \mathbf{Ax})$$

$$\therefore Z_D(u) = 10x_1 + 5x_2 + 8x_3 + 7x_4 + u(40 - 6x_1 - 5x_2 - 4x_3 - 6x_4)$$

Subject to

$\mathbf{Dx} \leq \mathbf{e}$,

which is

$3x_1 + x_2 \leq 15$

$x_1 + x_2 \leq 10$

$x_3 + 2x_4 \leq 10.$

Step 3. Assume $U^k = 1$ for $k = 0$. Then

Maximize

$$Z_D(u^k) = 10x_1 + 5x_2 + 8x_3 + 7x_4 + 40 - 6x_1 - 5x_2 - 4x_3 + 6x_4$$
$$= 4x_1 + 4x_3 + x_4 + 40,$$

such that $\mathbf{Dx} \leq \mathbf{e}$.

9.11 Conclusions

This chapter discusses a Lagrangian problem in which the complicated constraints were replaced with a penalty term in the objective function involving the amount of violation constraints and their dual variables. The Lagrangian relaxation concept and setting method were first discussed in Sections 9.2 through 9.4. A comparison with an LP-based bound was presented in Section 9.5, followed by an improved relaxation concept. Practical applications, such as for power systems, were presented in Sections 9.7 to 9.9.

The Lagrange relaxation technique has been presented in this chapter. Simplex method relaxes or eliminates complex constraints by adding them

to the master problem thereby enabling the independent solution to many subproblems.

The procedure for solving nonlinear programming optimization problems is shown in this chapter. The coupling constraints are handling them as additional constraints. It stems from Dantzig–Wolf description method for solving large linear problems. The algorithm involves the determination of λ value and controls the control and assorted state variables to be optimized. Application of Lagrangian relaxation (LR) to large-scale problems is generally applicable to security constrained economic dispatch OPF problem. Schemes to carry out implementation of Lagrange relaxation in UC is for scheduling for real-time application are identified in [10].

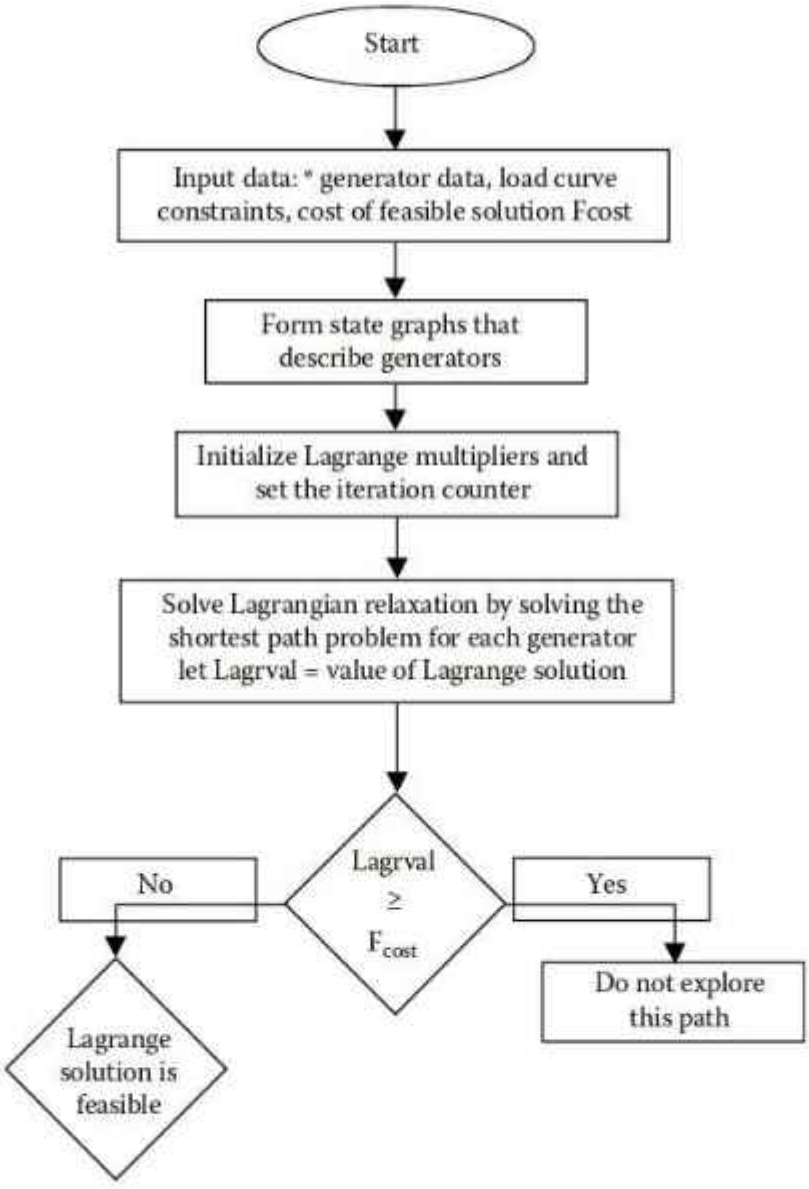


FIGURE 9.6
Algorithm for Lagrangian method.

9.12 Problem Set

PROBLEM 9.1

Maximize $z = 4x_1 + x_2 + 6x_3$

Subject to

$3x_1 + 2x_2 + 4x_3 \leq 17$

$x_1 \leq 2$

$x_3 \leq 2$

$x_1 \geq 1$

$x_2 \geq 1$

$x_3 \leq 1.$

PROBLEM 9.2

Maximize $z = 3x_1 + 5x_2$

Subject to

$3x_1 + 2x_2 \leq 18$

$x_1 \leq 4$

$x_j \leq 0.$

PROBLEM 9.3

Apply the principle to the following problem.

Maximize $z = 6x_1 + 7x_2 + 3x_3 + 5x_4 + x_5 + x_6$

Subject to

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 50$

$x_1 + x_2 \leq 10$

$x_2 \leq 8$

$5x_3 + x_4 \leq 12$

$x_5 + x_6 \leq 5$

$x_5 + 5x_6 \leq 50$

$x_j \leq 0.$

PROBLEM 9.4

Solve the following problem using the LR algorithm.

Maximize $z = 10x_1 + 2x_2 + 4x_3 + x_4$

Subject to

$x_1 + 4x_2 - x_3 \geq 8$

$2x_1 + x_2 + x_3 \geq 2$

$3x_1 + x_4 + x_5 \geq 4$

$x_1 + 2x_4 + x_5 \geq 10$

$x_j \leq 0.$

PROBLEM 9.5

Solve the following problem using the LR algorithm.

Maximize $P = 4x_1 + 5x_2 + 5x_3 + 4x_4$

Subject to

$2x_1 + 2x_2 + 3x_3 + 4x_4 \leq 7$

$x_1 - x_2 + x_3 - x_4 \leq 0$

$x_i \in \{0,1\}$

PROBLEM 9.6

Apply the LR algorithm to

Maximize $f = (0.3x_1^2 + 10)u_1 + (0.4x_2^2 + 14)u_2$

Subject to

$-x_1u_1 - x_2u_2 + 9.5 = 0$

$0 \leq x_1 \leq 0$

$0 \leq x_2 \leq 12$

$u_i \in \{0,1\}$

References

1. Fisher, M. L., The Lagrangian relaxation method for solving integer programming problems, *Management Science*, 27(1), 1981, 1–18.
2. Geoffrion, A. M., Lagrangian relaxation and its uses in integer programming, *Mathematical Programming Study*, 2, 1974, 82–114.

3. Graves, S. C., Using Lagrangian techniques to solve hierarchical production planning problems, *Management Science*, 28(3), 1982, 260–275.
4. Graves, S. C. and Lamar, B. W., An integer programming procedure for assembly system design problems, *Operations Research*, 31(3), 1983, 522–545.
5. Held, M. H., Wolfe, P., and Crowder, H. D., Validation of subgradient optimization, *Mathematical Programming*, 6(1), 1974, 62–88.
6. Muckstadt, J. A. and Koenig, S., An application of Lagrangian relaxation to scheduling in power generation systems, *Operations Research*, 25(3), 1977.
7. Shepardson, F. and Marsten, R. E., A Lagrangian relaxation algorithm for the two duty period scheduling problem, *Management Science*, 26, 1980, 274–281.
8. Shepardson, F. and Marsten, R. E., Solving a distribution problem with side constraints, *European Journal of Operations Research*, 6(1), 1981, 61–66.
9. Mulvey, J., Reducing the US Treasury's taxpayer data base by optimization, *Interfaces*, 10(5), 1980, 101–112.
10. Santos, A. and Costa, G. R. M., Optimal-power-flow solution by Newton's method applied to an augmented Lagrangian function, *IEE Proceedings—Generation, Transmission and Distribution*, 142(1), 1995, 33–36.



Dr. Bishaljit Paul presently is working as Assistant Professor of EE department at Narula Institute of Technology, Agarpara.
His research interests are Electricity Power Markets, Power Congestion Management Techniques, and Power System Optimization, Uncertainty Decision Making and Stochastic Programming.



Ms. Dipu Mistry is currently associated as an Assistant Professor of Narula Institute of Technology which is situated at Agarpara near Kolkata since 2008 in the Electrical Engineering Department. She has got her M.Tech degree in Material Engineering in the year of 2007 from Jadavpur University (J.U.) from the Department of Metellurgy .Her main subject of interest is Electrical machine, Power System Operation & Control sysyem. Now she is doing her PhD in the area of Electricity Power Market & scheduling of power despatch through renewable energy resources.



Ms. Priyanjali Mukherjee completed her M.Tech Degree in Control and Instrumentation from Netaji Subhash Engineering College in the year 2011.
She has in her credit more than 10 papers which are published in IEEE, IET conferences, book chapters in Springer Nature. She has a teaching experience of nearly twelve years. Presently she is working as Assistant Professor of EE department at Narula Institute of Technology, Agarpara. Her research interests are Electricity Power Markets, Power Congestion Management Techniques, and Power System Optimization, Uncertainty Decision Making and Stochastic Programming.



Ms. Reshmi Chandra completed her HS(CBSE,New Delhi) in 2001. She received BTech Degree in Electrical Engineering from MAKAUT,WB in 2007.
She completed her MTech from MAKAUT,WB in 2012.She received an award of University Topper in MTech(EDPS).
She has a teaching experience of nearly nine years. Presently she is working as Assistant Professor of EE Department at Abacus Institute of Engineering and Management,Magra.
Her research interest are Power system optimisation,Electricity power market, Uncertainty Decision Making. She is persuing PhD from NIT Mizoram.



Narula Institute of Technology is a premier educational institute in Kolkata. It is a part of JIS Educational Initiatives. Today, it ranks among the top private engineering college in Kolkata, West Bengal. Most of its eligible programs are NBA accredited. It has received several accolades and rankings from industry leaders. Some of them are NIRF, CAREERS360, India Today, The Week, and Times of India. Narula Institute of Technology stands among the best private engineering colleges in Kolkata, West Bengal through its near-perfect adherence to global quality standards.

Learnet Publishing
We value, we create

ISBN 978-81-977844-1-5
9 788197 784415